

Feynman rules of higher-order poles in CHY construction

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ABSTRACT: In this paper, we generalize the integration rules for scattering equations to situations where higher-order poles are present. We describe the strategy to deduce the Feynman rules of higher-order poles from known analytic results of simple CHY-integrands, and propose the Feynman rules for single double pole and triple pole as well as duplex-double pole and triplex-double pole structures. We demonstrate the validation and strength of these rules by ample non-trivial examples.

KEYWORDS: Scattering Amplitudes, Differential and Algebraic Geometry

ARXIV EPRINT: [1604.07314](https://arxiv.org/abs/1604.07314)

¹The unconventional ordering is to let authors get proper recognition of contributions under the outdated practice in China.

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1 Introduction

In the last few years, a new formulation for tree-level amplitudes of massless theories has been presented by Cachazo, He and Yuan [CHY] in a series of papers [1–5]. The formula is given by

$$\mathcal{A}_n = \int \frac{(\prod_{i=1}^n dz_i)}{\text{vol}(\text{SL}(2, \mathbb{C}))} \Omega(\mathcal{E}) \mathcal{I}(z) = \int \frac{(\prod_{i=1}^n dz_i)}{d\omega} \Omega(\mathcal{E}) \mathcal{I}, \quad (1.1)$$

where z_i are puncture locations of n external particles in CP^1 and $d\omega = \frac{dz_r dz_s dz_t}{z_{rs} z_{st} z_{tr}}$ comes after we use the Möbius $\text{SL}(2, \mathbb{C})$ symmetry to fix the location of three of the variables z_r, z_s, z_t by the Faddeev-Popov method. The Ω is defined by

$$\Omega(\mathcal{E}) \equiv z_{ij} z_{jk} z_{ki} \prod_{a \neq i, j, k} \delta(\mathcal{E}_a), \quad (1.2)$$

where \mathcal{E}_a 's are the *scattering equations* defined as

$$\mathcal{E}_a \equiv \sum_{b \neq a} \frac{s_{ab}}{z_a - z_b} = 0, \quad a = 1, 2, \dots, n. \quad (1.3)$$

As in the formula (1.1), the CHY construction involves two parts: *solving the scattering equations* \mathcal{E}_a which are universal for all theories, and *formulating the CHY-integrand* \mathcal{I} for a given theory.

Although conceptually, the CHY approach is remarkable and very useful for many theoretical studies on the properties of scattering amplitudes, when applying to practical evaluation, one confronts the problem of solving scattering equations. The scattering equation problem can be related to solving a polynomial equation system of degree $(n-3)!$ [6]. It is well-known that when $n \geq 6$, in general there is no analytic way to solve it.¹ To deal with this problem, many methods have been proposed. In [9], using classical formulas of Vieta, which relates the sums of roots of polynomials to the coefficients of these polynomials, analytic expression can be obtained without solving roots explicitly. The view of algebraic geometry has been further developed in several works. In [10] the so-called companion matrix method from computational algebraic geometry, tailored for zero-dimensional ideals, to study the scattering equations has been proposed. In [11], the Bezoutian matrix method has been used to facilitate the calculation of amplitudes. In [12–14], the elimination theory has been exploited to great details for the group of scattering equations with $(n-3)$ variables. In [15, 16] an efficient method has been developed to evaluate the Yang-Mills theory.

Different from the algebraic geometry methods, two more generic algorithms have been proposed. In one approach [17], using known results for scalar ϕ^3 theory, one can iteratively decompose the *4-regular* graph determined by the corresponding CHY-integrand to building blocks related to ϕ^3 theory, Which ends the evaluation. In another approach [18, 19], by careful analysis of pole structures, the authors wrote down an integration rule, so that from the related CHY-integrand, one can read out contributions from the corresponding Feynman diagrams directly if there are only simple poles appear. The later is really an efficient tool, however when higher-order poles are presented in a CHY-integrand, it can not proceed furthermore because of lacking Feynman rules for them.

In this paper, we would like to improve the efficiency of solving scattering equations by generalizing *integration rule* from only simple poles to higher-order poles, in the spirit of [18, 19]. The key-point of our generalization is to realize that, even for higher-order poles, one can find properly-defined Feynman rules. Although in current stage, we are not able to derive these generalized Feynman rules for higher-order poles, we have provided substantial evidences to support our claim.

This paper is organized as follows. In section 2, we provide a review of integration rule for simple poles. In section 3, we state the Feynman rules for some higher-order poles, with discussion on how to deduce these rules. Followed in section 4, we provide ample examples to support the proposed rules. In the last section, discussion and conclusion are provided.

Note added: during the preparation of this paper, a new formalism named Λ -algorithm has been presented by Gomez in [20], which provides an alternative algorithm to deal with higher-order poles.

¹There is some discussion on the special solutions in four-dimension [7, 8].

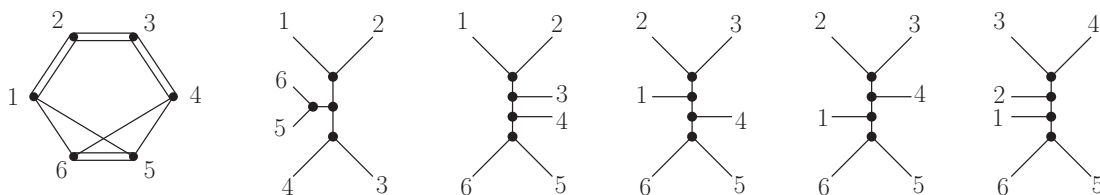


Figure 1. The left-most graph is an example of *4-regular* graph for certain CHY-integrand. The remaining five diagrams are contributing Feynman diagrams for amplitude of that CHY-integrand.

2 Review of integration algorithm

Before proposing the Feynman rules for higher-order poles, let us for reader's convenience review the *integration algorithm* presented in [18, 19, 21].² The whole algorithm can be roughly described as two parts: (1) given a CHY-integrand, find out the corresponding Feynman diagrams; (2) work out corresponding Feynman rules (or *generalized* Feynman rule while considering higher-order poles), with which write down the result directly for given Feynman diagrams.

For a n -point amplitude, the CHY-system is described by n complex variables $\{z_1, z_2, \dots, z_n\}$ before gauge fixing. The CHY-integrand, as a rational function of $z_{ij} \equiv z_i - z_j$, can be represented by a graph of n nodes with many lines connecting them. A factor z_{ij} in denominator(or numerator) can be represented by a solid line(or dashed line) connecting nodes i and j with an arrow pointing from i to j . A CHY-integrand which is invariant under Möbius transformation is then represented by a *4-regular* graph, i.e., a graph such that for each node, the number of attached solid lines minus the number of attached dashed lines is four. This is illustrated by an example with only simple poles as shown in figure 1. The corresponding CHY-integrand for this *4-regular* graph³ is

$$\frac{1}{(1, 2, 3, 4, 5, 6)(1, 2, 3, 4, 6, 5)} = \frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{56}^2 z_{45} z_{61} z_{46} z_{15}}. \quad (2.1)$$

The purpose of *integration algorithm* is to *read out* the result of a given CHY-integrand directly from the *4-regular* graph without solving any scattering equations. As described in [18, 19, 21], this is readily programmed for situations where only simple poles are present. Now we present a review on *integration algorithm*. For a n -point system, let $A = \{a_1, a_2, \dots, a_m\} \subset \{1, 2, \dots, n\}$, which is a subset of nodes. For each subset, let us define following **pole index**

$$\chi(A) := \mathbb{L}[A] - 2(|A| - 1), \quad (2.2)$$

where $\mathbb{L}[A]$ is the number⁴ of lines connecting these nodes inside A , and $|A|$ is the number of nodes. The condition

$$\chi(A) \geq 0 \quad (2.3)$$

²In these literatures, it is called *integration rule*. To not get confused with Feynman rules, we will call it *integration algorithm*.

³Throughout this paper we have omitted the arrows of lines in the graph for simplicity. Also when we read out result using *integration algorithm*, there is an overall sign one need to pay attention to.

⁴More accurately, it is the difference of number between solid and dashed lines.

will be called the **pole condition** for a given subset. Explicitly, each subset gives a non-zero pole contribution if and only if $\chi(A) \geq 0$. The pole will be in the form $\frac{1}{s_A^{\chi(A)+1}}$, where $s_A = (p_{a_1} + p_{a_2} + \dots + p_{a_m})^2$ for massless momentum p_i . Also, because of the momentum conservation, we will treat the subset A to be equivalent to its complement $\bar{A} = \{1, 2, \dots, n\} - A$, thus we can impose the length condition $2 \leq |A| \leq \lfloor \frac{n}{2} \rfloor$.

Once the pole structures of subsets are clear, we define the **compatible condition** for two subsets A_1, A_2 : two subsets are compatible to each other if one subset is completely contained inside another subset or the intersection of two subsets is empty.

Now we state the *integration algorithm*:

- (1) Given a CHY-integrand, draw the corresponding *4-regular* graph. For example, for the CHY-integrand (2.1) we draw the graph in figure 1.
- (2) Find all subsets A with $\chi(A) \geq 0$. For simple pole, we have $\chi(A) = 0$. For higher-order poles as considered in this paper, we have $\chi(A) > 0$.

For example (2.1) (see figure 1), since $n = 6$, we have $2 \leq |A| \leq 3$. For subsets with two nodes, it is easy to find that subsets $\{1, 2\}, \{2, 3\}, \{3, 4\}$ and $\{5, 6\}$ having $\chi = 0$, thus we should find simple poles $s_{12}, s_{23}, s_{34}, s_{56}$. For subsets with three nodes, $\{1, 2, 3\} \equiv \{4, 5, 6\}$ or $\{2, 3, 4\} = \{1, 5, 6\}$ having $\chi = 0$, thus we should also find poles s_{123}, s_{234} .

- (3) Find all maximum *compatible combinations*, i.e., the combination of subsets with largest number such that each pair in the combination is compatible.

Going back to our example, the combination $\{\{1, 2\}, \{2, 3\}, \{5, 6\}\}$ is not a compatible combination since $\{1, 2\}$ and $\{2, 3\}$ do not satisfy compatible condition. For this example, we can find five maximum compatible combinations

$$\begin{aligned} & \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}, \quad \{\{1, 2, 3\}, \{1, 2\}, \{5, 6\}\}, \quad \{\{1, 2, 3\}, \{2, 3\}, \{5, 6\}\}, \\ & \{\{2, 3, 4\}, \{2, 3\}, \{5, 6\}\}, \quad \{\{2, 3, 4\}, \{3, 4\}, \{5, 6\}\}. \end{aligned} \quad (2.4)$$

- (4) **Length condition for combination**: for each maximum combination with m subsets, it gives non-zero contribution if and only if $m = n - 3$. In the current example, all five combinations (2.4) give non-zero contributions.
- (5) For each compatible combination which has non-zero contribution, we can draw the corresponding Feynman diagram with only cubic vertices, and the propagators respecting the pole structures of compatible combination. Then we can use the (generalized) Feynman rule to write down results. For the above five compatible combinations given in (2.4), the corresponding Feynman diagrams are presented in figure 1. Using the rule for simple pole, we can write down result immediately as

$$\frac{1}{s_{12}s_{34}s_{56}} + \frac{1}{s_{123}s_{12}s_{56}} + \frac{1}{s_{123}s_{23}s_{56}} + \frac{1}{s_{234}s_{23}s_{56}} + \frac{1}{s_{234}s_{34}s_{56}}. \quad (2.5)$$

From above *integration algorithm*, one can see that all the five steps are in fact computer task except one, i.e., the determination of Feynman rules. For simple pole, the Feynman rule is nothing but the usual Feynman propagator $\frac{1}{s_A}$. For higher-order poles, the story becomes complicated, and they are the main context we would discuss in the following sections.

Before going on, there are some general remarks. The first is, why we believe there are Feynman rules for higher-order poles? The first hint comes from the *pinching* picture given in [22]. When discussing contributions with a given pole structure, it is found that we can group some nodes together to reduce a CHY-graph to two CHY-graphs with fewer nodes. Thus the contribution can be roughly written as $T_{s_A} \sim T_L \times \frac{1}{s_A} \times T_R$, where T_L, T_R have natural tree amplitude structure. Iterating the pinching procedure we can reduce a big CHY-graph to some building blocks. This picture is nothing but the familiar one when cutting a propagator in Feynman diagram to reduce a big diagram to two sub-diagrams, while the building block is nothing but the Feynman rules.

Keeping above picture in mind, now we present a more detailed discussion on the *pinching* operation. Let us divide n nodes to the subset $A \equiv \{z_{a_1}, z_{a_2}, \dots, z_{a_m}\}$ and its complement $B \equiv \{z_{a_{m+1}}, z_{a_2}, \dots, z_{a_n}\}$. The pinching corresponds to representing the whole subset by a new node and then adding this new node to the graph, which means we will get two new sets $\tilde{A} \equiv \{z_{a_1}, z_{a_2}, \dots, z_{a_m}, z_B\}$ and $\tilde{B} \equiv \{z_A, z_{a_{m+1}}, z_{a_2}, \dots, z_{a_n}\}$. Furthermore, all lines z_{ab} with $z_a \in A$ and $z_b \in B$ will be modified as follows: $z_{ab} \rightarrow z_{aB} = z_a - z_B$ in the new set \tilde{A} and $z_{ab} \rightarrow z_{Ab} = z_A - z_b$ in the new set \tilde{B} . Now we want each new set \tilde{A}, \tilde{B} to give legitimate CHY-graphs, i.e., *4-regular* graph. To see the condition, let us use $\mathbb{L}[A], \mathbb{L}[B], \mathbb{L}[A, B]$ to denote the number of lines inside the subset A, B and $\{A, B\}$ respectively. Then we have

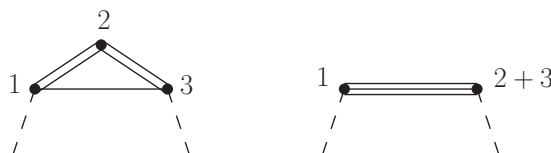
$$2n = \mathbb{L}[A] + \mathbb{L}[B] + \mathbb{L}[A, B], \quad 2(m+1) = \mathbb{L}[A] + \mathbb{L}[A, B], \quad 2(n-m+1) = \mathbb{L}[B] + \mathbb{L}[A, B]. \quad (2.6)$$

From these relations, we find

$$\mathbb{L}[A, B] = 4. \quad (2.7)$$

This condition leads to $\chi(A) = 0$, i.e., the subset contributes to simple pole $\frac{1}{s_A}$. It means we can only pinch nodes with simple poles.

Above pinching condition has an important implication. Using pinching, we can reduce a bigger CHY-graph to a smaller one as shown in following figure:



Iterating the procedure, we can reduce any graphs to a much simpler primary graph. No matter what the reduced graph looks like, its higher pole structure is invariant, i.e., the number and the degree of higher-order poles are not changing. The pinching picture has also inspired us to find the Feynman rule for a given higher-order pole with the strategy: *solving the simplest CHY-integrand containing such pole structure and then properly generalizing it!* This will be the approach we use in the paper.

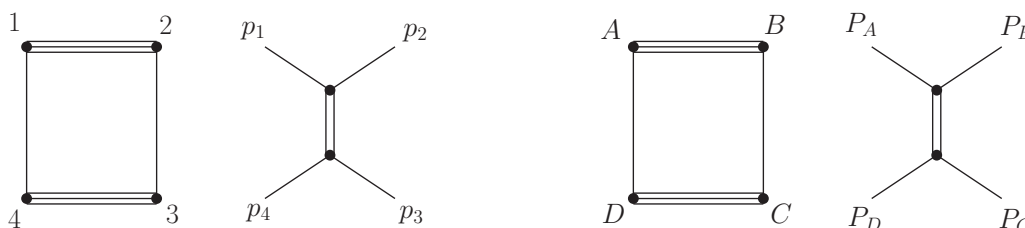


Figure 2. Left: the 4-regular graph of a 4-point CHY-integrand and its corresponding Feynman diagram. The *double-line* propagator denotes a double pole of this CHY-integrand. Right: labels of external momenta for the Feynman rule $\mathcal{R}_{\text{ule}}^I$ when legs are massive.

Our result of Feynman rules for higher-order poles has also revealed that, while the propagator of simple pole structure should be considered to be complete local (i.e., its Feynman rule $\frac{1}{s_A}$ depends only on the momentum flowing in between), the propagator of higher-order poles should be properly considered as *quasi-local*, i.e., its rule will depend also on the momentum as well as the types of poles that connecting to the four corners of propagators.

3 The Feynman rules for higher-order poles

As reviewed in section 2, the *integration algorithm* can be readily generalized to CHY-integrand with higher-order poles if we have corresponding Feynman rules for them. Once the Feynman rules of higher-order poles are in order, we can produce the results of any CHY-integrands instantly, following the standard five steps of *integration algorithm*.

In this section, we will provide Feynman rules for some higher-order poles. The general strategy of deducing these Feynman rules are described as follows. Firstly, we find out the simplest CHY-integrand which would contain the required higher-order pole, and obtain the analytic result by any reliable method. In many cases, one would find more than one Feynman diagram for a given CHY-integrand, and from them we need to isolate the contribution that only generated by Feynman rule of that higher-order pole. Then we generalize the result to generic situations by considering the symmetry structures of poles, the kinematics as well as when the external legs are massive. This involves *trial-and-error* by some other CHY-integrands to fix ambiguities. Once the result is formulated as a *rule*, then it can be applied in the computation of any CHY-integrands that containing corresponding higher-order poles.

3.1 The Feynman rule $\mathcal{R}_{\text{ule}}^I$ for a single double pole

The Feynman rule. For external legs as labeled in the right-most diagram of figure 2, the Feynman rule for a single double pole is formulated as

$$\mathcal{R}_{\text{ule}}^I[P_A, P_B, P_C, P_D] = \frac{2P_AP_C + 2P_BP_D}{2s_{AB}^2}, \quad (3.1)$$

where P with capital letters as subscripts denotes a sum of several massless legs, e.g., $P_A = \sum_{i=1}^m p_{a_i}$. Terms such as $2P_AP_B$ is understood to be the Minkowski products $2P_A \cdot$

$P_B = s_{AB} - P_A^2 - P_B^2$. We could notice the difference between $2P_AP_B$ and s_{AB} when external legs are massive.

Two remarks before proceed. Firstly, the Feynman diagram in figure 2 is blind with ordering. But for a given *4-regular* graph of CHY-integrand as shown in figure 2, especially paying attention to the lines connecting nodes A, D and nodes B, C , the resulting CHY-integrand do depends on the ordering of nodes, e.g., the ordering $\{A, B, C, D\}$ gives different result against ordering $\{A, B, D, C\}$. This point is very important when applying the rules of higher-order poles. One should be very careful when drawing the Feynman diagrams for CHY-integrands, especially for those legs connecting to the propagators of double poles. Secondly, different from the simple pole which is completely local, rule for higher-order pole depends not only on the total momentum $P_A + P_B$ flowing through the propagator, but also momenta P_A, P_B, P_C, P_D at four corners. Hence we call this propagator to be *quasi-local*. As we shall see very soon, when two propagators of double poles are connected by a vertex, the Feynman rule for this kind of pole structure will be different, i.e., we can not apply rule (3.1) to those two propagators, but consider them as a single object. This quasi-local property is the reason that deducing Feynman rules for higher-order poles is quite difficult.

Deducing the Feynman rule. As mentioned, the strategy is to find out the simplest CHY-integrand that containing the required higher-order pole structure and obtain the analytic result by any means. Thus we start from the simple four-point CHY-integrand,

$$\mathcal{I} = -\frac{1}{z_{12}^3 z_{23}^3 z_{34}^3 z_{41}^3}. \quad (3.2)$$

It can be trivially computed by directly solving scattering equations, with the result

$$\frac{s_{13}}{s_{12}^2}. \quad (3.3)$$

Now we try to reproduce above result by *integration algorithm* reviewed in previous section. We draw the *4-regular* graph for this CHY-integrand, as shown in figure 2. It is easy to see that there are only two independent subsets of nodes, i.e., $\{1, 2\}$ and $\{2, 3\}$. While $\chi[\{2, 3\}] = 1 - 2(2 - 1) = -1$, and $\chi[\{1, 2\}] = 3 - 2(2 - 1) = 1$, we conclude that the only compatible combination is $\{\underline{1}, 2\}$, where an underline is to emphasize that it corresponds to a double pole. There is only one Feynman diagram corresponding to this four-point CHY-integrand, as shown in figure 2 besides the *4-regular* graph, where we use a double line to denote the double pole.

In order to deduce the Feynman rule for this double pole, let us try to get some hints from the result (3.3). Note that the *4-regular* graph (originates from the CHY-integrand) apparently possesses symmetries under exchanging of nodes

$$\{1, 2, 3, 4\} \leftrightarrow \{4, 3, 2, 1\} \quad \text{and} \quad \{1, 2, 3, 4\} \leftrightarrow \{2, 1, 4, 3\}, \quad (3.4)$$

while under such exchanging, the kinematic variables are changed as

$$s_{13} \leftrightarrow s_{24}. \quad (3.5)$$

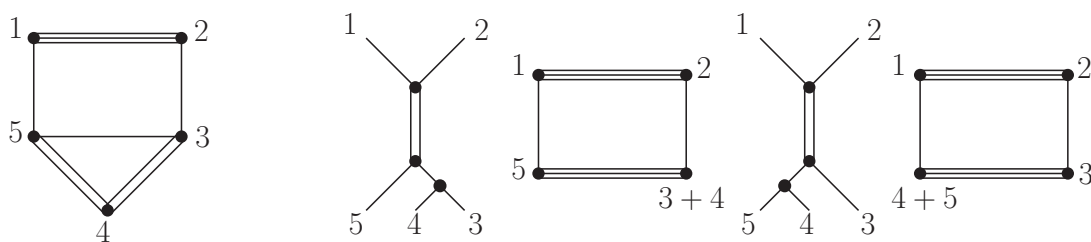


Figure 3. The 4 -regular graph of a given five-point CHY-integrand with a double pole, and two Feynman diagrams corresponding to this CHY-integrand. To indicate the ordering of two Feynman diagrams, we have drawn two pinched CHY-graphs.

It means that the numerator of (3.3) is not invariant under above exchanging. We would like to symmetrize it as⁵

$$\frac{s_{13} + s_{24}}{2s_{12}^2}. \quad (3.6)$$

However, expression (3.6) is not the final answer, since s_{12} is generally not equal to $2p_1 \cdot p_2$ if p_1, p_2 are massive. Since the numerator is not constrained by any considerations, both s_{ij} and $2p_i \cdot p_j$ could be possible candidates. This motivates us to consider the case when external momenta are massive as shown in the right-most diagram of figure 2, which will be a sum over several massless momenta for generic CHY-integrands. It forces us to determine a proper formulation for (3.6) that can be applied to the computation of more generic CHY-integrands. The denominator, since contains only physical poles, should be s_{AB} but not $2P_AP_B$. In this case, $s_{AB} = s_{CD}$ because of momentum conservation and we can take s_{AB}^2 as denominator. While for the numerator, a five-point example of CHY-integrand would suffice to tell us which one to choose. With *trial-and-error* method, one find that for massive case, we should use $2P_AP_B$ instead of s_{AB} in the numerator. Although it has no difference in the massless limit(i.e., the four-point CHY-integrand), it indeed introduces important corrections to guarantee the correctness for generic CHY-integrands. This leads us to formulate the Feynman rule of a single double pole as in (3.1).

An illustrative example. Let us illustrate the Feynman rule $\mathcal{R}_{\text{ule}}^{\text{I}}$ by a five-point CHY-integrand

$$-\frac{1}{z_{12}^3 z_{23} z_{34}^2 z_{45}^2 z_{53} z_{51}},$$

which is also studied in [17] by standard KLT construction. We can draw the 4 -regular graph for this CHY-integrand, as shown in figure 3. By counting the number of lines that connecting subsets of nodes, it is trivial to find that $\{1, 2\}$ contributes to a double pole, while $\{3, 4\}$, $\{4, 5\}$ contribute to a simple pole. The compatible combinations consist $5 - 3 = 2$ subsets of nodes that satisfying compatible condition, and here we have two

⁵In fact, if we want to get a totally symmetric expression under above exchanging, the denominator s_{12}^2 should be rewritten as $s_{12}s_{34}$. However, since we know poles are always in the form $((\sum_i p_i)^2)^a$, and by momentum conservation we have $s_{12} = s_{34}$. So we can keep denominator as s_{12}^2 . However, for numerator the story would be quite different.

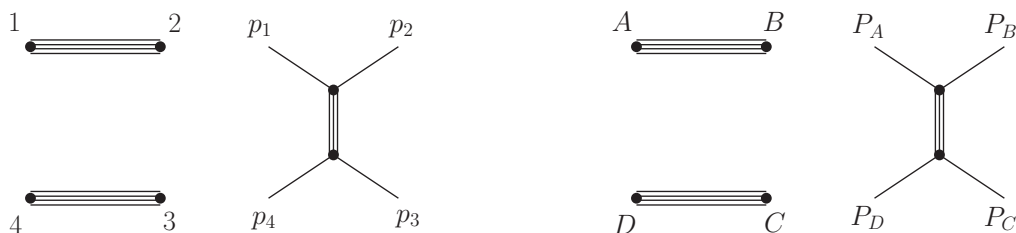


Figure 4. Left: the 4 -regular graph of a given CHY-integrand and its corresponding Feynman diagram. The *triple-line* propagator denotes a triple pole of this CHY-integrand. Right: labels of external momenta for the Feynman rule $\mathcal{R}_{\text{ule}}^{\text{II}}$ when legs are massive.

compatible combinations as

$$\{\{1, 2\}, \{3, 4\}\}, \quad \{\{1, 2\}, \{4, 5\}\} \quad (3.7)$$

From them we can draw two Feynman diagrams which have the required propagators as shown in figure 3. Note that for simple poles, the Feynman rule is not affected by the ordering of external legs attached to the propagator. For example, switching leg 3 and leg 4 of the first Feynman diagram in figure 3, we still get $\frac{1}{s_{34}}$ for the propagator. However, the Feynman rules of higher-order poles do depend on the ordering of external legs, as can be seen in the definition (3.1). The ordering of legs can be traced back to the four-point 4 -regular graph. For example, legs 3 and 4 in the first Feynman diagram are combined together and become a massive leg of the double pole, and the ordering of legs are determined by the ordering of nodes in the four-point 4 -regular graph besides the Feynman diagram. When applying the Feynman rule $\mathcal{R}_{\text{ule}}^{\text{I}}$ for the double pole, we should refer to the ordering marked by the 4 -regular graph. In order to use the Feynman rule of double pole directly in the Feynman diagram, we should draw the external legs in a definite ordering by using the four-point 4 -regular graph as assist. Hereafter, we will always draw Feynman diagrams with definite ordering of legs attached to the higher-order poles so that we can read out the rules directly from them, but keep in mind that this ordering of legs is determined by the 4 -regular graph.

Now we can write down the final result by summing contributions of two Feynman diagrams. It is

$$\begin{aligned} & \frac{1}{s_{34}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2\}, \{3, 4\}, \{5\}] + \frac{1}{s_{45}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2\}, \{3\}, \{4, 5\}] \\ &= \frac{1}{s_{34}} \frac{2p_1 p_{34} + 2p_2 p_5}{2s_{12}^2} + \frac{1}{s_{45}} \frac{2p_1 p_3 + 2p_2 p_{45}}{2s_{12}^2} = \frac{s_{25}}{s_{34} s_{12}^2} + \frac{s_{13}}{s_{45} s_{12}^2} - \frac{1}{s_{12}^2}. \end{aligned} \quad (3.8)$$

Note that the last term $-\frac{1}{s_{12}^2}$ in the result is in fact a correction term introduced by using $2P_A P_B$ for massive legs instead of s_{AB} in the Feynman rule. This result agrees with that in [17].

3.2 The Feynman rule $\mathcal{R}_{\text{ule}}^{\text{II}}$ for a single triple pole

The Feynman rule. For external legs as labeled in the right-most diagram of figure 4, the Feynman rule for a single triple pole is formulated as

$$\begin{aligned} \mathcal{R}_{\text{ule}}^{\text{II}}[P_A, P_B, P_C, P_D] & \quad (3.9) \\ &= \frac{(2P_AP_C)(2P_AP_D) + (2P_BP_C)(2P_BP_D) + (2P_CP_A)(2P_CP_B) + (2P_DP_A)(2P_DP_B)}{4s_{AB}^3} \\ & \quad - \frac{(P_A^2 - P_B^2)^2 + (P_C^2 - P_D^2)^2}{4s_{AB}^3} + \frac{2(P_A^2 + P_B^2)(P_C^2 + P_D^2)}{9 \cdot 4s_{AB}^3}. \end{aligned}$$

Note that terms in the second line are zero when all external legs are massless, so they can not be deduced from the result of four-point CHY-integrand. Nevertheless, it is very important in order to produce correct answer for general situations.

Deducing the Feynman rule. Again let us start from a simple four-point CHY-integrand, which contains this pole structure and is given by

$$\mathcal{I} = \frac{1}{z_{12}^4 z_{34}^4}. \quad (3.10)$$

The result can be trivially obtained by solving scattering equations as

$$\frac{s_{14}s_{13}}{s_{12}^3}. \quad (3.11)$$

Now we follow the *integration algorithm* for this example. In the *4-regular* graph as shown in figure 4, it is easy to see that $\chi[\{2, 3\}] = 0 - 2(2 - 1) = -2$ while $\chi[\{1, 2\}] = 4 - 2(2 - 1) = 2$, which means there is only one subset $\{\underline{1}, \underline{2}\}$ corresponding to a triple pole (a two-fold underline is introduced to emphasize it). So we can draw the Feynman diagram for this CHY-integrand, as shown in figure 4 besides the *4-regular* graph, where a triple line is introduced to denote the triple pole.

To deduce (but not derive!) the Feynman rule for a single triple pole from result (3.11), we follow the similar considerations as in previous subsection. Apparently, the *4-regular* graph is invariant under exchanging of nodes 1, 2 as well as exchanging of nodes 3, 4. It is also invariant under exchanging $i \rightarrow \text{Mod}[i + 2, 4]$. From these considerations, intuitively we can propose a symmetrization for (3.11) as

$$\begin{aligned} & \frac{s_{14}s_{13} + s_{24}s_{23} + s_{41}s_{42} + s_{31}s_{32}}{4s_{12}^3} \\ \rightarrow \mathcal{R}' &= \frac{(2P_AP_D)(2P_AP_C) + (2P_BP_D)(2P_BP_C) + (2P_DP_A)(2P_DP_B) + (2P_CP_A)(2P_CP_B)}{4s_{AB}^3}, \end{aligned} \quad (3.12)$$

using identities $s_{13} = s_{24}$, $s_{14} = s_{23}$, and also the rule shown above. However, when computing the five-point CHY-integrand as shown in figure 5, the above rule fails to produce correct answer. But another rule

$$\mathcal{R}'' = \frac{(2P_AP_D)(2P_AP_C) + (2P_BP_D)(2P_BP_C)}{2s_{AB}^3}, \quad (3.13)$$

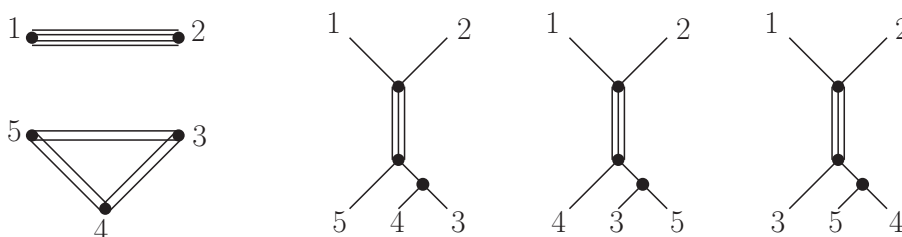


Figure 5. The 4 -regular graph of a given five-point CHY-integrand with a triple pole, and two Feynman diagrams corresponding to this CHY-integrand.

although not preserving the required symmetries, can produce correct answer. Anyway, we need a rule that preserving the symmetries of 4 -regular graph, and this motivates us to inspect the difference between above two expressions (3.12), (3.13). Notice that for the five-point CHY-integrand in figure 5, we have applied the rule under the condition that $P_A = p_1, P_B = p_2$ are massless, while P_C, P_D could be massive. After taking the difference of (3.13) and (3.12), we get

$$R'' - R' = -\frac{(P_C^2 - P_D^2)^2}{4s_{AB}^2}. \quad (3.14)$$

This indicates that \mathcal{R}' plus the extra term would produce correct result for CHY-integrand in figure 5. Considering symmetries of 4 -regular graph, it is reasonable to refine the rule as⁶

$$\mathcal{R}' - \frac{(P_A^2 - P_B^2)^2 + (P_C^2 - P_D^2)^2}{4s_{AB}^3}. \quad (3.15)$$

However, this is still not the complete rule. For CHY-integrand in figure 5, we do not confront situations that two or more external legs are massive. We found that, when both P_A, P_B or both P_C, P_D are on-shell, the rule (3.15) still works, otherwise it fails. This motive us to add another correction term, with the property that it becomes zero when $P_A^2 = P_B^2 = 0$ or $P_C^2 = P_D^2 = 0$, but non-zero when $P_A^2 = P_C^2 = 0$ or $P_B^2 = P_D^2 = 0$. An immediate candidate is $(P_A^2 + P_B^2)(P_C^2 + P_D^2)$, and after *trial-and-error*, we find that a correction term

$$\frac{2}{9} \frac{(P_A^2 + P_B^2)(P_C^2 + P_D^2)}{4s_{AB}^3} \quad (3.16)$$

would suffice to produce correct results for applying the rule of triple pole to all possible situations of massive legs. Then we formulated the Feynman rule $\mathcal{R}_{\text{ule}}^{\text{H}}$ for triple pole as shown in (3.9).

⁶We comment that, if only relying on the symmetry consideration, there would be many possible terms to be added in the rule, for example

$$\frac{(P_A^2 + P_B^2)^2 + (P_C^2 + P_D^2)^2}{4s_{AB}^3}, \quad \frac{(P_A^2 + P_B^2 + P_C^2 + P_D^2)^2}{4s_{AB}^3}, \quad \text{etc.}$$

So if there is no *trial-and-error* process with other CHY-integrands, it is in fact difficult to formulate a valid rule from these possible terms due to too many ambiguities.



Figure 6. Left: the 4 -regular graph of a given CHY-integrand and its corresponding Feynman diagram. The *double-line* propagator denotes a double pole of this CHY-integrand. Since two propagators of double poles are connected at one vertex, we define it as duplex-double pole. Right: labels of external momenta for the Feynman rule $\mathcal{R}_{\text{ule}}^{\text{III}}$ when external legs are massive.

An illustrative example. Let us illustrate the Feynman rule $\mathcal{R}_{\text{ule}}^{\text{II}}$ by a five-point CHY-integrand

$$-\frac{1}{z_{12}^4 z_{34}^2 z_{45}^2 z_{53}^2}.$$

The 4 -regular graph for this CHY-integrand is drawn in figure 5. By counting the number of lines connecting subsets of nodes, we find that $\{\underline{1, 2}\}$ is associated with a triple pole, while $\{3, 4\}, \{4, 5\}, \{5, 3\}$ are associated with simple poles. We need to select $5 - 3 = 2$ subsets to construct the compatible combinations, and there are three,

$$\{\{\underline{1, 2}\}, \{3, 4\}\}, \quad \{\{\underline{1, 2}\}, \{4, 5\}\}, \quad \{\{\underline{1, 2}\}, \{3, 5\}\}. \quad (3.17)$$

From them we can draw three Feynman diagrams, as shown in figure 5 besides 4 -regular graph. Note that in the definition of $\mathcal{R}_{\text{ule}}^{\text{II}}$, the exchanging $P_A \leftrightarrow P_B$ or $P_C \leftrightarrow P_D$ will not affect the rule (which is not true for $\mathcal{R}_{\text{ule}}^{\text{I}}$). So there is no definite ordering for external legs attached to the same end of triple pole propagator. Then it is straightforward to write down the result by applying Feynman rules of triple pole as

$$\begin{aligned} & \frac{1}{s_{34}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{3, 4\}, \{5\}] + \frac{1}{s_{45}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{4, 5\}, \{3\}] + \frac{1}{s_{35}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{3, 5\}, \{4\}] \\ &= \frac{(2p_1 p_{34})(2p_1 p_5) + (2p_2 p_{34})(2p_2 p_5) + (2p_{34} p_1)(2p_{34} p_2) + (2p_5 p_1)(2p_5 p_2) - ((p_3 + p_4)^2)^2}{4s_{12}^3 s_{34}} \\ &+ \frac{(2p_1 p_{45})(2p_1 p_3) + (2p_2 p_{45})(2p_2 p_3) + (2p_{45} p_1)(2p_{45} p_2) + (2p_3 p_1)(2p_3 p_2) - ((p_4 + p_5)^2)^2}{4s_{12}^3 s_{45}} \\ &+ \frac{(2p_1 p_{35})(2p_1 p_4) + (2p_2 p_{35})(2p_2 p_4) + (2p_{35} p_1)(2p_{35} p_2) + (2p_4 p_1)(2p_4 p_2) - ((p_3 + p_5)^2)^2}{4s_{12}^3 s_{35}} \\ &= \frac{s_{15} s_{25}}{s_{12}^3 s_{34}} + \frac{s_{13} s_{23}}{s_{12}^3 s_{45}} + \frac{s_{14} s_{24}}{s_{12}^3 s_{35}} + \frac{1}{s_{12}^2}. \end{aligned}$$

3.3 The Feynman rule $\mathcal{R}_{\text{ule}}^{\text{III}}$ for duplex-double pole

The Feynman rule. For external legs as labeled in the right-most diagram of figure 6, the Feynman rule for duplex-double pole is formulated as

$$\mathcal{R}_{\text{ule}}^{\text{III}}[P_A, P_B, P_E, P_C, P_D] = \frac{(2P_A P_D)(2P_B P_C) - (2P_A P_C)(2P_B P_D)}{s_{AB}^2 s_{CD}^2}$$

$$-\frac{(P_E^2)(2P_AP_D + 2P_BP_C - 2P_AP_C - 2P_BP_D)}{4s_{AB}^2 s_{CD}^2}. \quad (3.18)$$

Notice that in the five-point CHY-integrand, we have $P_E^2 = p_3^2 = 0$, which means that the second term in fact can not be deduced from five-point result. But it is non-zero for generic CHY-integrands, and is crucial in order to produce the correct answer.

Deducing the Feynman rule. Although the Feynman diagram in figure 6 contains only double pole, these two propagators of double poles are connected at a vertex. Practically, we find it impossible to use the Feynman rule $\mathcal{R}_{\text{ule}}^I$ separately for each double pole, and should treat them as a single object which we call duplex-double pole. In order to deduce the Feynman rule for this duplex-double pole, let us start from a simple five-point CHY-integrand,

$$\mathcal{I} = -\frac{1}{z_{12}^3 z_{23} z_{34} z_{45}^3 z_{53} z_{31}}. \quad (3.19)$$

Although it is not as trivial as the four-point CHY-integrands, we can still compute it by solving scattering equations directly. After some simplification, we can write the result as

$$\frac{s_{15}s_{24} - s_{14}s_{25}}{s_{12}^2 s_{45}^2}. \quad (3.20)$$

Of course we can rewrite it in other equivalent forms by using momentum conservation, but at this point let us just take this expression.

The *4-regular* graph of this CHY-integrand is shown in figure 6. It is easy to figure out that the possible subsets of nodes that contributing to poles are $\{\underline{1}, 2\}$ and $\{4, \underline{5}\}$. Since $\chi[\{1, 2\}] = \chi[\{4, 5\}] = 1$, both of them are double poles. They can only form one compatible combination $\{\{\underline{1}, 2\}, \{4, \underline{5}\}\}$, and from which we draw the Feynman diagrams as shown besides the *4-regular* graph in figure 6. This means that we should deduce the Feynman rule for duplex-double pole from result (3.20).

Again considering the symmetries among nodes of the *4-regular* graph(or the CHY-integrand), clearly it is

$$\begin{aligned} &\text{anti-symmetric under} \quad \{1, 2\} \leftrightarrow \{2, 1\} \quad \text{or} \quad \{4, 5\} \leftrightarrow \{5, 4\}, \\ &\text{symmetric under} \quad \{1, 2, 3, 4, 5\} \leftrightarrow \{4, 5, 3, 1, 2\} \quad \text{or} \quad \{1, 2, 3, 4, 5\} \leftrightarrow \{2, 1, 3, 5, 4\}. \end{aligned}$$

By inspecting the kinematic variables, it is easy to see that the result (3.20) already possesses above anti-symmetries as well as symmetries, so it is not necessary to further symmetrize it. With experiences of previous Feynman rules of higher-order poles, it is straightforwardly to propose a Feynman rule

$$\frac{(2P_AP_D)(2P_BP_C) - (2P_AP_C)(2P_BP_D)}{s_{AB}^2 s_{CD}^2} \quad (3.21)$$

for massive legs as shown in the right-most diagram of figure 6. However, it is not yet the complete rule. After *trial-and-error*, we find that the second term in $\mathcal{R}_{\text{ule}}^{\text{III}}$ (3.18) should be included. In the five-point CHY-integrand, since $P_E^2 = p_3^2 = 0$, it vanishes and we can never deduce it from (3.20). But it is crucial in order to produce the correct result for other generic CHY-integrands, and we will show this in section 4.4 with examples containing mixed types of higher order poles.

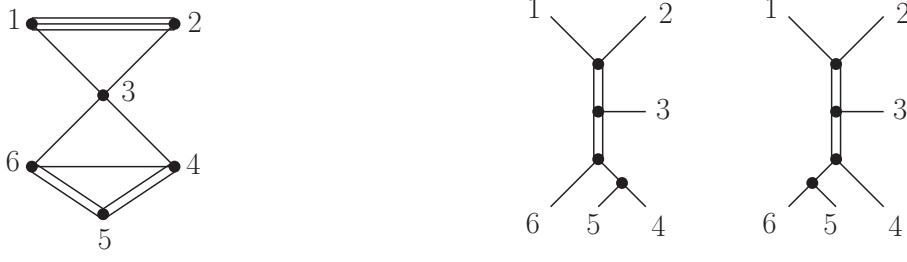


Figure 7. The 4-regular graph of a given CHY-integrand with duplex-double poles, and two Feynman diagrams corresponding to this CHY-integrand.

An illustrative example. We will illustrate the Feynman rule $\mathcal{R}_{\text{ule}}^{\text{III}}$ by a six-point CHY-integrand

$$-\frac{1}{z_{12}^3 z_{23} z_{34} z_{45}^2 z_{56}^2 z_{64} z_{63} z_{31}}. \quad (3.22)$$

The 4-regular graph of this CHY-integrand is shown in figure 7. Following the *integration algorithm*, firstly we list all the subsets of nodes that contribute to either simple pole or higher-order poles, as

$$\{1, 2\}, \quad \{4, 5, 6\}, \quad \{4, 5\}, \quad \{5, 6\}.$$

Secondly, we need to select $6 - 3 = 3$ subsets out of above four subsets to construct compatible combinations. They are given by

$$\{\{1, 2\}, \{4, 5, 6\}, \{4, 5\}\}, \quad \{\{1, 2\}, \{4, 5, 6\}, \{5, 6\}\}.$$

Immediately we draw two Feynman diagrams as shown in figure 7, which have the required propagators according to compatible combinations respectively. We have intentionally labeled the external legs in certain ordering such that we can directly read out the Feynman rules from the diagram. Then according to the rules, we get the result

$$\begin{aligned} & \frac{1}{s_{45}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2\}, \{3\}, \{4, 5\}, \{6\}] + \frac{1}{s_{56}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2\}, \{3\}, \{4\}, \{5, 6\}] \\ &= \frac{(2p_1 p_{45})(2p_2 p_6) - (2p_1 p_6)(2p_2 p_{45})}{s_{45} s_{12}^2 s_{456}^2} + \frac{(2p_1 p_4)(2p_2 p_{56}) - (2p_1 p_{56})(2p_2 p_4)}{s_{56} s_{12}^2 s_{456}^2} \\ &= \frac{s_{145} s_{26} - s_{16} s_{245}}{s_{45} s_{12}^2 s_{456}^2} + \frac{s_{14} s_{256} - s_{156} s_{24}}{s_{56} s_{12}^2 s_{456}^2} + \frac{s_{16} - s_{26} + s_{24} - s_{14}}{s_{12}^2 s_{456}^2}. \end{aligned} \quad (3.23)$$

The last term is in fact the correction term introduced by using $2P_A P_B$ instead of s_{AB} for massive legs in the Feynman rules. This result is confirmed numerically.

3.4 The Feynman rule $\mathcal{R}_{\text{ule}}^{\text{IX}}$ for triplex-double pole

We have presented the Feynman rules $\mathcal{R}_{\text{ule}}^{\text{I}}, \mathcal{R}_{\text{ule}}^{\text{II}}, \mathcal{R}_{\text{ule}}^{\text{III}}$ for double pole, triple pole and duplex-double pole. They are reasonably simple. But when going to the higher-point CHY-integrands, new structures of poles would appear. Deducing rules for them could be very difficult, and the rules themselves could be complicated. To illustrate, let us present another Feynman rule $\mathcal{R}_{\text{ule}}^{\text{IX}}$ for a pole structure that first appear in six-point CHY-integrand.

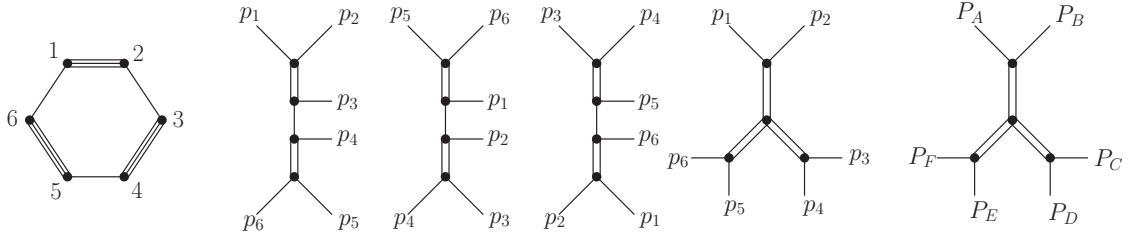


Figure 8. Left-most: the 4-regular graph of a given CHY-integrand. Middle-four: its corresponding four Feynman diagrams. The *double-line* propagator denotes a double pole of this CHY-integrand. In the fourth Feynman diagram, three double poles are connected at one vertex, and we define it as triplex-double pole. Right-most: labels of external momenta for the Feynman rule $\mathcal{R}_{\text{ule}}^{\text{IX}}$ when legs are massive.

The Feynman rule. For external legs as labeled in the right-most diagram of figure 8, the Feynman rule for triplex-double pole is formulated as (for a better presentation, we define the *stripped* Mandelstam variables $\tilde{s}_{AB} = 2P_AP_B$ as well as $\tilde{s}_{ABC} = 2P_AP_B + 2P_AP_C + 2P_BP_C$ ⁷)

$$\begin{aligned} & \mathcal{R}_{\text{ule}}^{\text{IX}}[P_A, P_B, P_C, P_D, P_E, P_F] \\ &= \left(\frac{\mathcal{R}_{11}}{8s_{AB}^2 s_{CD}^2} + \frac{\mathcal{R}_{12}}{8s_{CD}^2 s_{EF}^2} + \frac{\mathcal{R}_{13}}{8s_{EF}^2 s_{AB}^2} \right) + \left(\frac{\mathcal{R}_{21}}{2s_{AB}^2 s_{CD} s_{EF}} + \frac{\mathcal{R}_{22}}{2s_{CD}^2 s_{EF} s_{AB}} + \frac{\mathcal{R}_{23}}{2s_{EF}^2 s_{AB} s_{CD}} \right) \\ &+ \left(\frac{\mathcal{R}_{31}}{2s_{AB}^2 s_{CD}^2 s_{EF}} + \frac{\mathcal{R}_{32}}{2s_{CD}^2 s_{EF}^2 s_{AB}} + \frac{\mathcal{R}_{33}}{2s_{EF}^2 s_{AB}^2 s_{CD}} \right) + \frac{\mathcal{R}_4}{2s_{AB}^2 s_{CD}^2 s_{EF}^2} + \frac{1}{s_{AB} s_{CD} s_{EF}} \\ &- (P_A^2 + P_B^2 + P_C^2 + P_D^2 + P_E^2 + P_F^2) \left(\frac{1}{4s_{AB}^2 s_{CD}^2} + \frac{1}{4s_{AB}^2 s_{EF}^2} + \frac{1}{4s_{CD}^2 s_{EF}^2} \right), \end{aligned} \quad (3.24)$$

where

$$\mathcal{R}_{11} = 2(\tilde{s}_{EC} + \tilde{s}_{FB} - \tilde{s}_{EB} - \tilde{s}_{FC}) - (\tilde{s}_{ABC} + \tilde{s}_{BCD} + \tilde{s}_{CDE} + \tilde{s}_{DEF} + \tilde{s}_{EFA} + \tilde{s}_{FAB}), \quad (3.25)$$

$$\mathcal{R}_{12} = 2(\tilde{s}_{AE} + \tilde{s}_{BD} - \tilde{s}_{AD} - \tilde{s}_{BE}) - (\tilde{s}_{ABC} + \tilde{s}_{BCD} + \tilde{s}_{CDE} + \tilde{s}_{DEF} + \tilde{s}_{EFA} + \tilde{s}_{FAB}), \quad (3.26)$$

$$\mathcal{R}_{13} = 2(\tilde{s}_{CA} + \tilde{s}_{DF} - \tilde{s}_{CF} - \tilde{s}_{DA}) - (\tilde{s}_{ABC} + \tilde{s}_{BCD} + \tilde{s}_{CDE} + \tilde{s}_{DEF} + \tilde{s}_{EFA} + \tilde{s}_{FAB}), \quad (3.27)$$

$$\mathcal{R}_{21} = \tilde{s}_{AF} + \tilde{s}_{BC} + \tilde{s}_{AC} + \tilde{s}_{BF} - (\tilde{s}_{ACE} + \tilde{s}_{BDF}), \quad (3.28)$$

$$\mathcal{R}_{22} = \tilde{s}_{CB} + \tilde{s}_{DE} + \tilde{s}_{CE} + \tilde{s}_{DB} - (\tilde{s}_{ACE} + \tilde{s}_{BDF}), \quad (3.29)$$

$$\mathcal{R}_{23} = \tilde{s}_{ED} + \tilde{s}_{FA} + \tilde{s}_{EA} + \tilde{s}_{FD} - (\tilde{s}_{ACE} + \tilde{s}_{BDF}), \quad (3.30)$$

and

$$\begin{aligned} \mathcal{R}_{31} &= \tilde{s}_{BC}(\tilde{s}_{ED} + \tilde{s}_{FA} - \tilde{s}_{EC} - \tilde{s}_{FB}) \\ &+ (\tilde{s}_{CA}\tilde{s}_{DE} + \tilde{s}_{BD}\tilde{s}_{AF} - \tilde{s}_{BF}\tilde{s}_{CA} - \tilde{s}_{CE}\tilde{s}_{BD}) + (\tilde{s}_{CA} - \tilde{s}_{BD})(\tilde{s}_{CE} - \tilde{s}_{BF}), \end{aligned} \quad (3.31)$$

$$\begin{aligned} \mathcal{R}_{32} &= \tilde{s}_{DE}(\tilde{s}_{AF} + \tilde{s}_{BC} - \tilde{s}_{AE} - \tilde{s}_{BD}) \\ &+ (\tilde{s}_{EC}\tilde{s}_{FA} + \tilde{s}_{DF}\tilde{s}_{CB} - \tilde{s}_{DB}\tilde{s}_{EC} - \tilde{s}_{EA}\tilde{s}_{DF}) + (\tilde{s}_{EC} - \tilde{s}_{DF})(\tilde{s}_{EA} - \tilde{s}_{DB}), \end{aligned} \quad (3.32)$$

$$\mathcal{R}_{33} = \tilde{s}_{FA}(\tilde{s}_{CB} + \tilde{s}_{DE} - \tilde{s}_{CA} - \tilde{s}_{DF})$$

⁷When A, B , etc., contains only one massless leg, $\tilde{s} = s$, otherwise they differs. For example, if $A = \{1, 2\}, B = \{3\}$, $s_{AB} = (p_1 + p_2 + p_3)^2 = 2p_1p_2 + 2p_1p_3 + 2p_2p_3$ but $\tilde{s}_{AB} = 2p_{12}p_3 = 2p_1p_3 + 2p_2p_3$.

$$+ (\tilde{s}_{AE}\tilde{s}_{BC} + \tilde{s}_{FB}\tilde{s}_{ED} - \tilde{s}_{FD}\tilde{s}_{AE} - \tilde{s}_{AC}\tilde{s}_{FB}) + (\tilde{s}_{AE} - \tilde{s}_{FB})(\tilde{s}_{AC} - \tilde{s}_{FD}), \quad (3.33)$$

$$\begin{aligned} \mathcal{R}_4 = & \tilde{s}_{BC}(\tilde{s}_{CE}\tilde{s}_{EA} + \tilde{s}_{BF}\tilde{s}_{FD}) + \tilde{s}_{DE}(\tilde{s}_{EA}\tilde{s}_{AC} + \tilde{s}_{DB}\tilde{s}_{BF}) + \tilde{s}_{FA}(\tilde{s}_{AC}\tilde{s}_{CE} + \tilde{s}_{FD}\tilde{s}_{DB}) \\ & + \tilde{s}_{BC}\tilde{s}_{DE}(\tilde{s}_{EA} + \tilde{s}_{BF}) + \tilde{s}_{DE}\tilde{s}_{FA}(\tilde{s}_{AC} + \tilde{s}_{DB}) + \tilde{s}_{FA}\tilde{s}_{BC}(\tilde{s}_{CE} + \tilde{s}_{FD}) + 2\tilde{s}_{BC}\tilde{s}_{DE}\tilde{s}_{FA}. \end{aligned} \quad (3.34)$$

Note again that the last term in (3.24) vanishes for six-point CHY-integrand, and it can not be deduced from there. However it indeed contributes when any of external legs is sum of more than one massless momenta. We need this correction term to produce correct answer.

Deducing the Feynman rule. When applying *integration algorithm* to a six-point CHY-integrand

$$\frac{1}{z_{12}^3 z_{34}^3 z_{56}^3 z_{23} z_{45} z_{61}}, \quad (3.35)$$

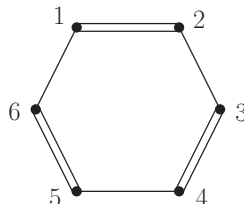
with its *4-regular* graph drawn in figure 8, we find six subsets of nodes

$$\{\underline{1}, 2\}, \quad \{3, \underline{4}\}, \quad \{5, \underline{6}\}, \quad \{1, 2, 3\}, \quad \{2, 3, 4\}, \quad \{3, 4, 5\} \quad (3.36)$$

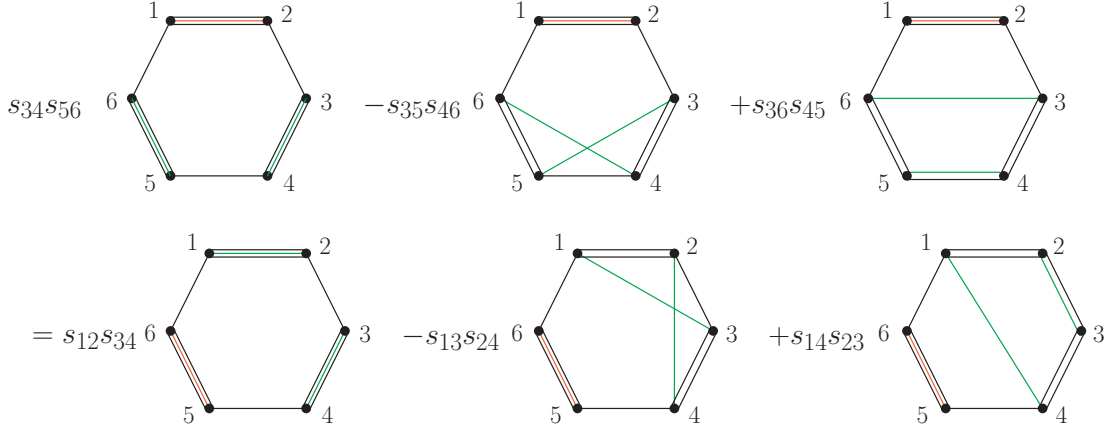
that will contribute to poles, of which three are associated to double poles. From them we can construct four compatible combinations, and the corresponding four Feynman diagrams are shown in figure 8. The first three Feynman diagrams can be computed by rule for simple poles and $\mathcal{R}_{\text{ule}}^{\text{I}}$. However, for the last Feynman diagram, three double poles are connected to one vertex, and we can neither use rule for double pole nor rule for duplex-double pole to compute it. Therefore, we need to create a rule $\mathcal{R}_{\text{ule}}^{\text{IX}}$ for this pole structure, which we call triplex-double pole. It has $1/(s_{12}^2 s_{34}^2 s_{56}^2)$ pole structure.

Recall that when deducing Feynman rules $\mathcal{R}_{\text{ule}}^{\text{I}}, \mathcal{R}_{\text{ule}}^{\text{II}}, \mathcal{R}_{\text{ule}}^{\text{III}}$ for double, triple and duplex-double poles, we always start from known results of the simplest CHY-integrands which exactly contain one Feynman diagram of that pole structure. However, the six-point CHY-integrand (3.35) is the first one that appears the triplex-double pole, and it has four Feynman diagrams. It is impossible to find another CHY-integrand that contains only one Feynman diagram, of which is exactly the triplex-double pole. So in order to deduce the Feynman rule $\mathcal{R}_{\text{ule}}^{\text{IX}}$, we have no choice but play with the result of CHY-integrand (3.35).

In order to get the result of (3.35), one can use the Pfaffian identity, as is already shown in [18, 19]. Starting from a template of *3-regular* graph,



we have the Pfaffian identity, for example,



The first diagram in the first line and second line is the one we want to compute, and now we can instead compute the remaining four in the identity, which could be CHY-integrands with only simple poles or higher-order poles. In paper [18, 19], one need more Pfaffian identities if some of the remaining CHY-integrands still possess higher-order poles. However, since now we have the Feynman rule for double pole, the remaining four CHY-integrands can be instantly computed, without reducing to those only with simple poles. Anyway, we can phrase the result as

$$\mathcal{A}_{\text{triple}} = T_1 + T_2, \quad (3.37)$$

where

$$T_1 = \frac{s_{13}s_{46}}{s_{12}^2 s_{56}^2 s_{123}} + \frac{s_{15}s_{24}}{s_{34}^2 s_{56}^2 s_{234}} + \frac{s_{26}s_{35}}{s_{12}^2 s_{34}^2 s_{345}}, \quad (3.38)$$

which contains simple poles s_{123} , s_{234} , s_{345} , and

$$\begin{aligned} T_2 = & -\frac{s_{13}}{s_{12}s_{34}s_{56}^2} + \frac{s_{36}s_{13}}{s_{12}^2 s_{34}s_{56}^2} + \frac{s_{46}s_{13}}{s_{12}^2 s_{34}s_{56}^2} + \frac{s_{123}}{s_{12}s_{34}s_{56}^2} + \frac{s_{36}s_{45}s_{134}}{s_{12}^2 s_{34}^2 s_{56}^2} - \frac{s_{26}}{s_{12}s_{34}s_{56}^2} \\ & + \frac{s_{45}}{s_{12}s_{34}s_{56}^2} + \frac{s_{26}s_{35}}{s_{12}^2 s_{34}s_{56}^2} + \frac{s_{26}s_{36}}{s_{12}^2 s_{34}s_{56}^2} - \frac{s_{36}s_{45}}{s_{12}^2 s_{34}s_{56}^2} + \frac{s_{35}s_{46}}{s_{12}^2 s_{34}s_{56}^2} - \frac{s_{46}}{s_{12}s_{34}s_{56}^2} \\ & + \frac{s_{35}s_{46}}{s_{12}^2 s_{34}s_{56}^2} + \frac{s_{36}s_{46}}{s_{12}^2 s_{34}s_{56}^2} - \frac{s_{36}s_{123}}{s_{12}^2 s_{34}s_{56}^2} + \frac{s_{15}s_{24}}{s_{12}s_{34}s_{56}^2} - \frac{s_{14}s_{25}}{s_{12}s_{34}s_{56}^2} - \frac{s_{45}s_{134}}{s_{12}s_{34}s_{56}^2} - \frac{s_{35}s_{46}s_{134}}{s_{12}^2 s_{34}s_{56}^2}. \end{aligned} \quad (3.39)$$

On the other hand, summing over all four Feynman diagrams in figure 8, we should also get the result

$$\mathcal{A}_{\text{triple}} = F_1 + F_2 + F_3 + F_4, \quad (3.40)$$

where

$$\begin{aligned} F_1 &= \frac{1}{s_{123}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3\}, \{4, 5, 6\}] \mathcal{R}_{\text{ule}}^I[\{1, 2, 3\}, \{4\}, \{5\}, \{6\}] \\ &= \frac{s_{13}s_{46}}{s_{12}^2 s_{56}^2 s_{123}} - \frac{2s_{13} + 2s_{46} - s_{123}}{4s_{12}^2 s_{56}^2}, \end{aligned} \quad (3.41)$$

$$\begin{aligned} F_2 &= \frac{1}{s_{234}} \mathcal{R}_{\text{ule}}^I[\{5\}, \{6\}, \{1\}, \{2, 3, 4\}] \mathcal{R}_{\text{ule}}^I[\{5, 6, 1\}, \{2\}, \{3\}, \{4\}] \\ &= \frac{s_{15}s_{24}}{s_{34}^2 s_{56}^2 s_{234}} - \frac{2s_{15} + 2s_{24} - s_{234}}{4s_{34}^2 s_{56}^2}, \end{aligned} \quad (3.42)$$

and

$$\begin{aligned}
 F_3 &= \frac{1}{s_{345}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{3\}, \{4\}, \{5\}, \{6, 1, 2\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{3, 4, 5\}, \{6\}, \{1\}, \{2\}] \\
 &= \frac{s_{35}s_{26}}{s_{12}^2 s_{34}^2 s_{345}} - \frac{2s_{35} + 2s_{26} - s_{345}}{4s_{12}^2 s_{34}^2}.
 \end{aligned} \tag{3.43}$$

F_4 should be generated by the rule of triplex-double pole we want to deduce.

From results F_1, F_2, F_3 it is immediately to see that, the terms T_1 , i.e., with simple poles $s_{123}, s_{234}, s_{345}$, is really computed by F_1, F_2, F_3 . This is consistent with the fact that the triplex-double pole can not produce simple poles $s_{123}, s_{234}, s_{345}$. The equality

$$T_1 + T_2 = F_1 + F_2 + F_3 + F_4 \tag{3.44}$$

then provides us an expression for F_4 , or the Feynman rule $\mathcal{R}_{\text{ule}}^{\text{IX}}$, with only double poles of s_{12}, s_{34}, s_{56} . We can rephrase the expression as

$$\begin{aligned}
 F_4 &= \frac{s_{12}^2(s_{15} + s_{24} - s_{14} - s_{25})}{4s_{12}^2 s_{34}^2 s_{56}^2} + \frac{s_{34}^2(s_{13} + s_{46} - s_{14} - s_{36})}{4s_{12}^2 s_{34}^2 s_{56}^2} + \frac{s_{56}^2(s_{26} + s_{35} - s_{25} - s_{36})}{4s_{12}^2 s_{34}^2 s_{56}^2} \\
 &\quad - \frac{(s_{12}^2 + s_{34}^2 + s_{56}^2)(s_{123} + s_{234} + s_{345})}{4s_{12}^2 s_{34}^2 s_{56}^2} \\
 &\quad + \frac{s_{12}s_{56}(s_{15} + s_{26})}{s_{12}^2 s_{34}^2 s_{56}^2} + \frac{s_{34}s_{56}(s_{35} + s_{46})}{s_{12}^2 s_{34}^2 s_{56}^2} + \frac{s_{12}s_{34}(s_{13} + s_{24})}{s_{12}^2 s_{34}^2 s_{56}^2} \\
 &\quad - \frac{(s_{12}s_{15} + s_{26}s_{56})s_{123}}{s_{12}^2 s_{34}^2 s_{56}^2} - \frac{(s_{34}s_{46} + s_{35}s_{56})s_{234}}{s_{12}^2 s_{34}^2 s_{56}^2} - \frac{(s_{12}s_{24} + s_{13}s_{34})s_{345}}{s_{12}^2 s_{34}^2 s_{56}^2} \\
 &\quad - \frac{(s_{12}s_{34} + s_{12}s_{56} + s_{34}s_{56})s_{135}}{s_{12}^2 s_{34}^2 s_{56}^2} + \frac{s_{123}s_{234}s_{345}}{s_{12}^2 s_{34}^2 s_{56}^2}.
 \end{aligned} \tag{3.45}$$

To deduce the Feynman rule from such a complicated result is quite difficult. While considering the symmetries, apparently, the 4-regular graph in figure 8 is invariant under exchanging

$$i \rightarrow \text{Mod}[i + 2, 6] \quad \text{as well as} \quad \{1, 2, 3, 4, 5, 6\} \leftrightarrow \{2, 1, 6, 5, 4, 3\}, \tag{3.46}$$

and similarly other exchanging of legs. We should reformulate F_4 in such a way that these symmetries are still kept. However, there are too many possibilities to reformulate F_4 due to the momentum conservation and massless conditions, and we have no strategy to choose the one that can be generalized as rule. There is another very important difference here compared to the previous rules. For $\mathcal{R}_{\text{ule}}^{\text{I}}, \mathcal{R}_{\text{ule}}^{\text{II}}, \mathcal{R}_{\text{ule}}^{\text{III}}$, it is possible to reformulate the numerator of the known results such that the s variable of higher-order pole do not present in the numerator. So we do not need to consider the possibility that whether s in the numerator would cancel a factor of higher-order pole or not. Although not explicitly mentioned, this is a guide line to deduce the Feynman rules $\mathcal{R}_{\text{ule}}^{\text{I}}, \mathcal{R}_{\text{ule}}^{\text{II}}, \mathcal{R}_{\text{ule}}^{\text{III}}$. However, here it seems impossible to completely eliminate s_{12}, s_{34}, s_{56} dependence in the numerator of F_4 by momentum conservation, thus it is also a possibility that they would cancel a factor of double poles. All these ambiguities makes things difficult. By extensive *trial-and-error* method with hints from a seven-point CHY-integrand, we finally formulate a valid Feynman rule $\mathcal{R}_{\text{ule}}^{\text{IX}}$ for triplex-double pole as presented in (3.24), and at least it works for the examples we checked up to nine points.

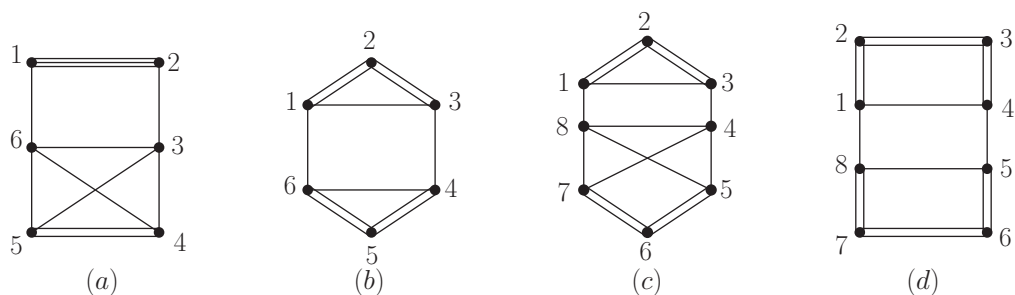


Figure 9. The 4-regular graphs of supporting examples with only double poles and simple poles.

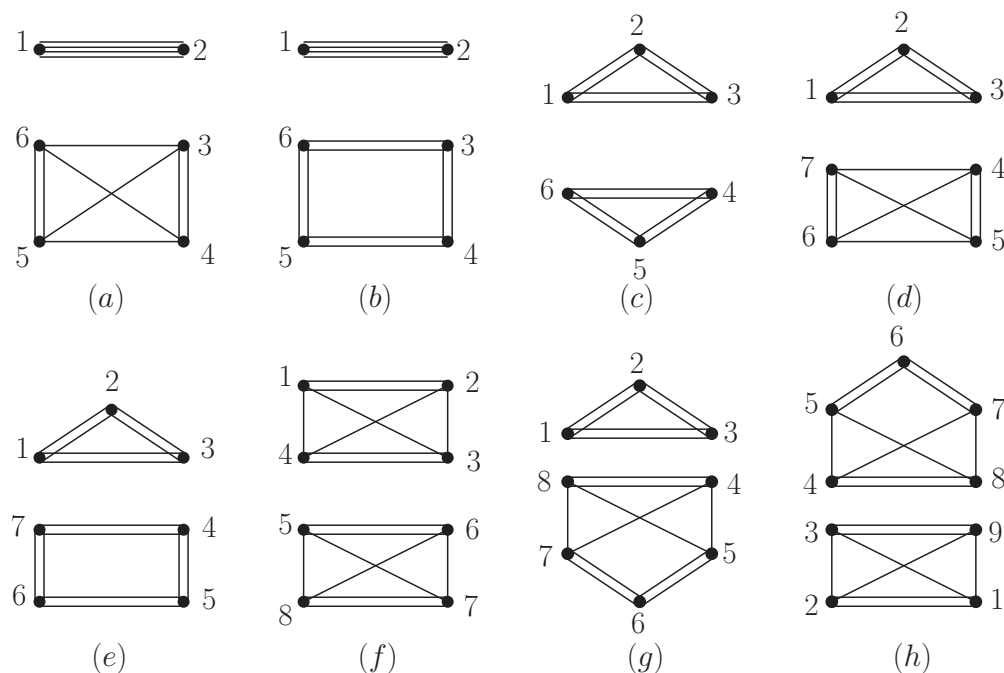


Figure 10. The 4-regular graphs of supporting examples with only triple poles and simple poles.

4 Supporting examples

The Feynman rules $\mathcal{R}_{\text{ule}}^{\text{I}}$, $\mathcal{R}_{\text{ule}}^{\text{II}}$, $\mathcal{R}_{\text{ule}}^{\text{III}}$, $\mathcal{R}_{\text{ule}}^{\text{IX}}$ are deduced but not derived, and their validation is supported by ample examples. The *integration algorithm*, as reviewed in section 2, involves nothing but selecting subsets of nodes according to pole condition, constructing compatible combinations and performing the pattern matching to apply the rules. All these steps can be implemented by a few lines of codes in **Mathematica**. One is able to produce the analytic result within seconds, while if trying to obtain a result from solving scattering equations, even numerically it would take for example two or more hours for nine-point CHY-integrands in a laptop (this is the reason why we do not check the rules by ten or even more point examples).

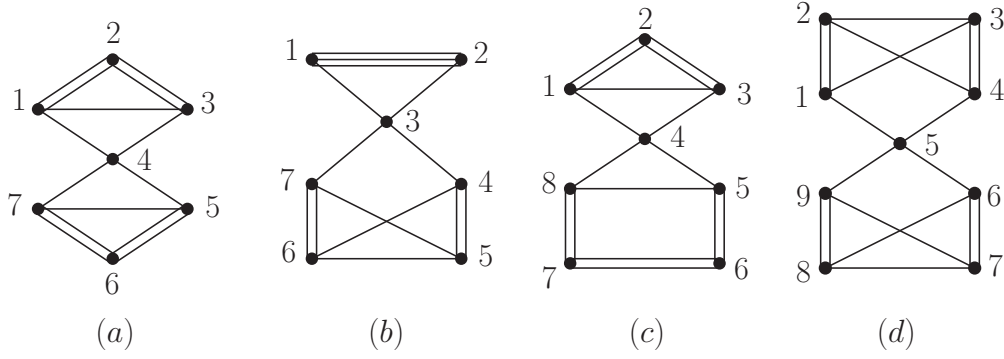


Figure 11. The 4 -regular graphs of supporting examples with only duplex-double poles and simple poles.

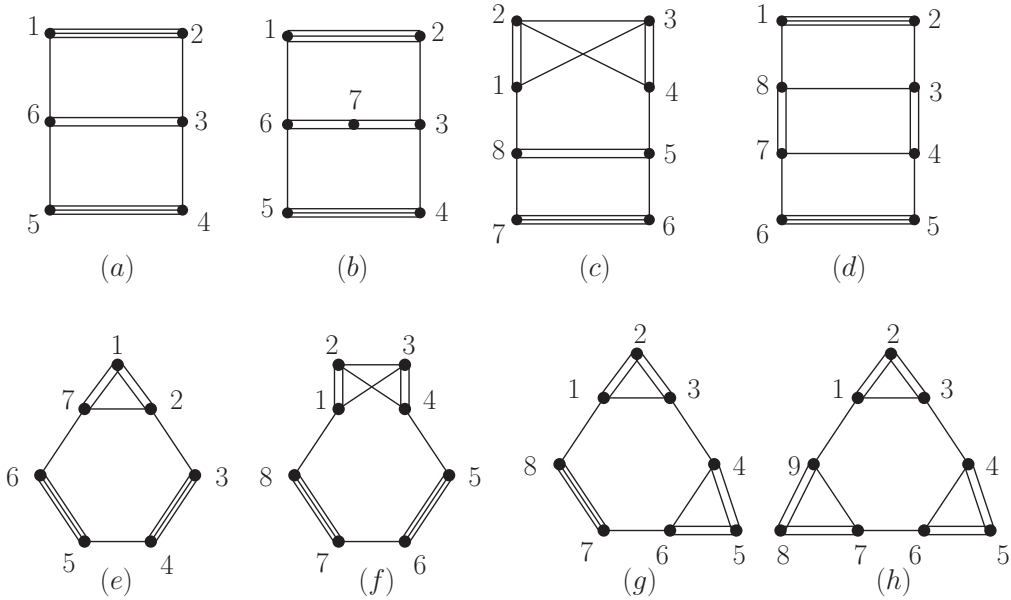


Figure 12. The 4 -regular graphs of supporting examples with mixed types of higher-order poles and simple poles. CHY-integrands in the first line contain double poles and duplex-double poles, while those in the second line contain double poles and triplex-double poles.

For those who are not familiar with *integration algorithm*, we present here the results of various CHY-integrands by rules of higher-order poles, and all have been checked numerically. For reader's convenience, we collect all the 4 -regular graphs of examples in figure 9, figure 10, figure 11 and figure 12. In case that some readers are specially interested in certain CHY-integrand, they can find the 4 -regular graph in these figures and jump to corresponding subsection for details.

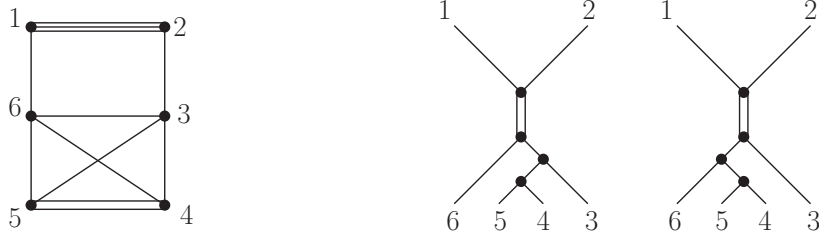


Figure 13. The 4-regular graph of a given CHY-integrand, and its two corresponding Feynman diagrams.

4.1 Examples with only double poles and simple poles

In this subsection, we will examine the Feynman rule $\mathcal{R}_{\text{ule}}^I$ of double pole by four examples in figure 9.

Example figure 9.a: this is a six-point example with CHY-integrand

$$\frac{1}{z_{12}^3 z_{45}^2 z_{23} z_{16} z_{36} z_{35} z_{46} z_{34} z_{56}}. \quad (4.1)$$

The possible subsets for poles are $\{1, 2\}$, $\{4, 5\}$, $\{3, 4, 5\}$, $\{4, 5, 6\}$, so from them we can only construct two compatible combinations of subsets as

$$\{\{1, 2\}, \{4, 5\}, \{3, 4, 5\}\}, \quad \{\{1, 2\}, \{4, 5\}, \{4, 5, 6\}\}.$$

From these combinations we can draw two corresponding Feynman diagrams as shown in figure 13, and using $\mathcal{R}_{\text{ule}}^I$ we can read out

$$\begin{aligned} & \frac{1}{s_{45} s_{345}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3, 4, 5\}, \{6\}] + \frac{1}{s_{45} s_{456}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3\}, \{4, 5, 6\}] \\ &= \frac{2p_1 p_{345} + 2p_2 p_6}{2s_{12}^2 s_{45} s_{345}} + \frac{2p_1 p_3 + 2p_2 p_{456}}{2s_{12}^2 s_{45} s_{456}} = \frac{s_{13}}{s_{12}^2 s_{45} s_{456}} + \frac{s_{26}}{s_{12}^2 s_{45} s_{345}} - \frac{1}{s_{12}^2 s_{45}}. \end{aligned} \quad (4.2)$$

Example figure 9.b: this is a six-point example with CHY-integrand

$$\frac{1}{z_{12}^2 z_{23}^2 z_{45}^2 z_{56}^2 z_{13} z_{34} z_{16} z_{46}}. \quad (4.3)$$

The possible subsets for poles are $\{1, 2, 3\}$, $\{1, 2\}$, $\{2, 3\}$, $\{4, 5\}$, $\{5, 6\}$, so four compatible combinations of subsets can be found as

$$\begin{aligned} & \{\{1, 2, 3\}, \{1, 2\}, \{4, 5\}\}, & \{\{1, 2, 3\}, \{1, 2\}, \{5, 6\}\}, \\ & \{\{1, 2, 3\}, \{2, 3\}, \{4, 5\}\}, & \{\{1, 2, 3\}, \{2, 3\}, \{5, 6\}\}. \end{aligned}$$

From these combinations we can draw four corresponding Feynman diagrams as shown in figure 14. Using the rule $\mathcal{R}_{\text{ule}}^I$, we get

$$\frac{1}{s_{12} s_{45}} \mathcal{R}_{\text{ule}}^I[\{1, 2\}, \{3\}, \{4, 5\}, \{6\}] + \frac{1}{s_{12} s_{56}} \mathcal{R}_{\text{ule}}^I[\{1, 2\}, \{3\}, \{4\}, \{5, 6\}] \quad (4.4)$$

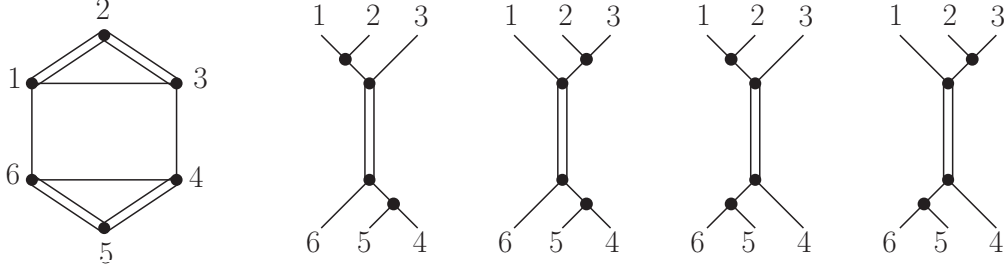


Figure 14. The 4-regular graph of a given CHY-integrand, and its four corresponding Feynman diagrams.

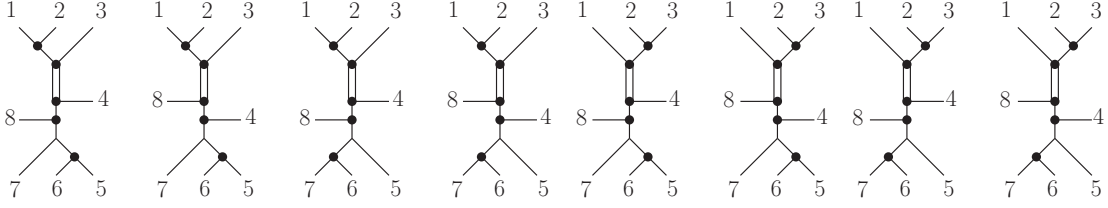


Figure 15. The eight Feynman diagrams corresponding to the 4-regular graph figure 9.c.

$$\begin{aligned}
 & + \frac{1}{s_{23}s_{45}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2, 3\}, \{4, 5\}, \{6\}] + \frac{1}{s_{23}s_{56}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2, 3\}, \{4\}, \{5, 6\}] \\
 & = \frac{2p_{12}p_{45} + 2p_3p_6}{2s_{123}^2 s_{12}s_{45}} + \frac{2p_{12}p_4 + 2p_3p_{56}}{2s_{123}^2 s_{12}s_{56}} + \frac{2p_{14}p_{45} + 2p_{23}p_6}{2s_{123}^2 s_{23}s_{45}} + \frac{2p_{14}p_4 + 2p_{23}p_{56}}{2s_{123}^2 s_{23}s_{56}} \\
 & = \frac{s_{14}}{s_{23}s_{56}s_{123}^2} + \frac{s_{124}}{s_{12}s_{56}s_{123}^2} + \frac{s_{145}}{s_{23}s_{45}s_{123}^2} - \frac{1}{s_{12}s_{123}^2} - \frac{1}{s_{23}s_{123}^2} - \frac{1}{s_{45}s_{123}^2} + \frac{s_{36}}{s_{12}s_{45}s_{123}^2} - \frac{1}{s_{56}s_{123}^2}.
 \end{aligned}$$

Example figure 9.c: this is an eight-point example with CHY-integrand

$$\frac{1}{z_{12}^2 z_{23}^2 z_{56}^2 z_{67}^2 z_{13} z_{18} z_{34} z_{48} z_{45} z_{78} z_{47} z_{58}}. \quad (4.5)$$

The possible subsets for poles are $\{1, 2, 3\}$, $\{4, 5, 6, 7\}$, $\{5, 6, 7, 8\}$, $\{5, 6, 7\}$, $\{1, 2\}$, $\{2, 3\}$, $\{5, 6\}$, $\{6, 7\}$, so we can construct eight compatible combinations of subsets as

$$\begin{aligned}
 & \{\{1, 2, 3\}, \{4, 5, 6, 7\}, \{5, 6, 7\}, \{5, 6\}, \{1, 2\}\}, \quad \{\{1, 2, 3\}, \{4, 5, 6, 7\}, \{5, 6, 7\}, \{6, 7\}, \{1, 2\}\}, \\
 & \{\{1, 2, 3\}, \{4, 5, 6, 7\}, \{5, 6, 7\}, \{5, 6\}, \{2, 3\}\}, \quad \{\{1, 2, 3\}, \{4, 5, 6, 7\}, \{5, 6, 7\}, \{6, 7\}, \{2, 3\}\}, \\
 & \{\{1, 2, 3\}, \{5, 6, 7, 8\}, \{5, 6, 7\}, \{5, 6\}, \{1, 2\}\}, \quad \{\{1, 2, 3\}, \{5, 6, 7, 8\}, \{5, 6, 7\}, \{6, 7\}, \{1, 2\}\}, \\
 & \{\{1, 2, 3\}, \{5, 6, 7, 8\}, \{5, 6, 7\}, \{5, 6\}, \{2, 3\}\}, \quad \{\{1, 2, 3\}, \{5, 6, 7, 8\}, \{5, 6, 7\}, \{6, 7\}, \{2, 3\}\}.
 \end{aligned}$$

From them we can draw eight Feynman diagrams as shown in figure 15, According to the rule $\mathcal{R}_{\text{ule}}^I$, the result is given by

$$\begin{aligned}
 & \frac{1}{s_{12}s_{567}s_{4567}} \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^I[\{1, 2\}, \{3\}, \{4, 5, 6, 7\}, \{8\}] \\
 & + \frac{1}{s_{12}s_{567}s_{5678}} \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^I[\{1, 2\}, \{3\}, \{4\}, \{5, 6, 7, 8\}]
 \end{aligned}$$

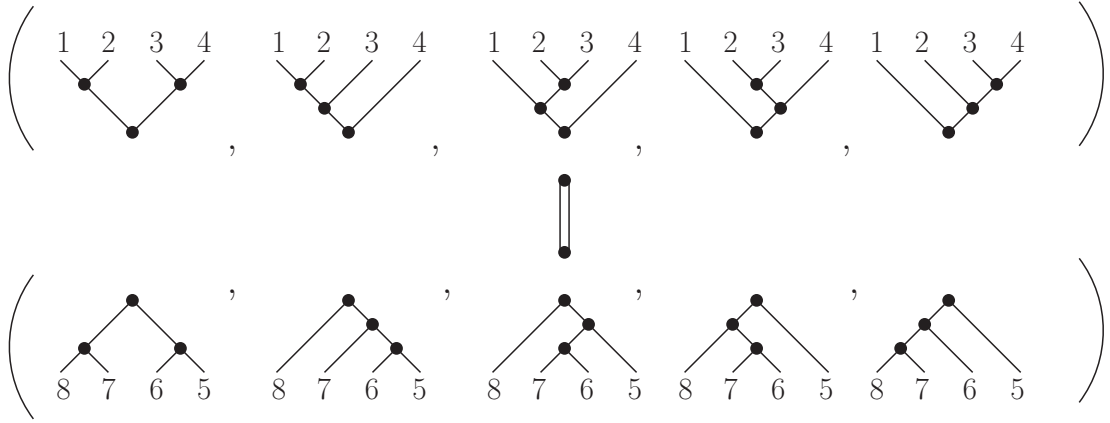


Figure 16. The 25 Feynman diagrams corresponding to the 4-regular graph figure 9.d.

$$\begin{aligned}
 & + \frac{1}{s_{23}s_{567}s_{4567}} \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2, 3\}, \{4, 5, 6, 7\}, \{8\}] \\
 & + \frac{1}{s_{23}s_{567}s_{5678}} \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2, 3\}, \{4\}, \{5, 6, 7, 8\}].
 \end{aligned}$$

Example figure 9.d: this is an eight-point example with CHY-integrand

$$\frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{56}^2 z_{67}^2 z_{78}^2 z_{45} z_{58} z_{18} z_{14}}. \quad (4.6)$$

The possible subsets for poles are $\{1, 2, 3, 4\}$, $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{5, 6, 7\}$, $\{6, 7, 8\}$, $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{5, 6\}$, $\{6, 7\}$, $\{7, 8\}$, so we can construct 25 compatible combinations of subsets as

$$\begin{aligned}
 & \{\{1, 2, 3, 4\}, \{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}, \\
 & \{\{1, 2, 3, 4\}, \{1, 2, 3\}, [\{1, 2\} \text{ or } \{2, 3\}], \{5, 6\}, \{7, 8\}\}, \\
 & \{\{1, 2, 3, 4\}, \{2, 3, 4\}, [\{2, 3\} \text{ or } \{3, 4\}], \{5, 6\}, \{7, 8\}\}, \\
 & \{\{1, 2, 3, 4\}, \{1, 2\}, \{3, 4\}, \{5, 6, 7\}, [\{5, 6\} \text{ or } \{6, 7\}]\}, \\
 & \{\{1, 2, 3, 4\}, \{1, 2\}, \{3, 4\}, \{6, 7, 8\}, [\{6, 7\} \text{ or } \{7, 8\}]\}, \\
 & \{\{1, 2, 3, 4\}, \{1, 2, 3\}, \{5, 6, 7\}, [\{1, 2\} \text{ or } \{2, 3\}], [\{5, 6\} \text{ or } \{6, 7\}]\}, \\
 & \{\{1, 2, 3, 4\}, \{1, 2, 3\}, \{6, 7, 8\}, [\{1, 2\} \text{ or } \{2, 3\}], [\{6, 7\} \text{ or } \{7, 8\}]\}, \\
 & \{\{1, 2, 3, 4\}, \{2, 3, 4\}, \{5, 6, 7\}, [\{2, 3\} \text{ or } \{3, 4\}], [\{5, 6\} \text{ or } \{6, 7\}]\}, \\
 & \{\{1, 2, 3, 4\}, \{2, 3, 4\}, \{6, 7, 8\}, [\{2, 3\} \text{ or } \{3, 4\}], [\{6, 7\} \text{ or } \{7, 8\}]\}.
 \end{aligned}$$

From them we can draw 25 Feynman diagrams as shown in figure 16, which immediately leads to the final result by Feynman rule $\mathcal{R}_{\text{ule}}^{\text{I}}$ as

$$\frac{1}{s_{12}s_{34}s_{56}s_{78}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}]$$

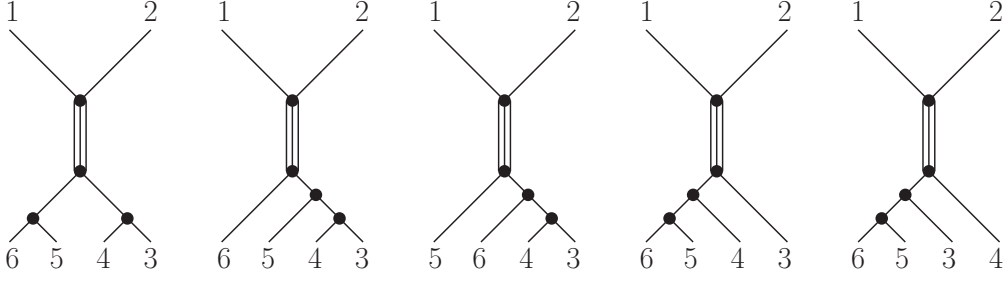


Figure 17. The five Feynman diagrams corresponding to the 4-regular graph figure 10.a.

$$\begin{aligned}
 & + \frac{1}{s_{123}s_{56}s_{78}} \left(\frac{1}{s_{12}} + \frac{1}{s_{23}} \right) \mathcal{R}_{\text{ule}}^{\text{I}}[\{1, 2, 3\}, \{4\}, \{5, 6\}, \{7, 8\}] \\
 & + \frac{1}{s_{234}s_{56}s_{78}} \left(\frac{1}{s_{23}} + \frac{1}{s_{34}} \right) \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2, 3, 4\}, \{5, 6\}, \{7, 8\}] \\
 & + \frac{1}{s_{12}s_{34}s_{567}} \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{I}}[\{1, 2\}, \{3, 4\}, \{5, 6, 7\}, \{8\}] \\
 & + \frac{1}{s_{12}s_{34}s_{678}} \left(\frac{1}{s_{67}} + \frac{1}{s_{78}} \right) \mathcal{R}_{\text{ule}}^{\text{I}}[\{1, 2\}, \{3, 4\}, \{5\}, \{6, 7, 8\}] \\
 & + \frac{1}{s_{123}s_{567}} \left(\frac{1}{s_{12}} + \frac{1}{s_{23}} \right) \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{I}}[\{1, 2, 3\}, \{4\}, \{5, 6, 7\}, \{8\}] \\
 & + \frac{1}{s_{123}s_{678}} \left(\frac{1}{s_{12}} + \frac{1}{s_{23}} \right) \left(\frac{1}{s_{67}} + \frac{1}{s_{78}} \right) \mathcal{R}_{\text{ule}}^{\text{I}}[\{1, 2, 3\}, \{4\}, \{5\}, \{6, 7, 8\}] \\
 & + \frac{1}{s_{234}s_{567}} \left(\frac{1}{s_{23}} + \frac{1}{s_{34}} \right) \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2, 3, 4\}, \{5, 6, 7\}, \{8\}] \\
 & + \frac{1}{s_{234}s_{678}} \left(\frac{1}{s_{23}} + \frac{1}{s_{34}} \right) \left(\frac{1}{s_{67}} + \frac{1}{s_{78}} \right) \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2, 3, 4\}, \{5\}, \{6, 7, 8\}]. \quad (4.7)
 \end{aligned}$$

4.2 Examples with only triple poles and simple poles

In this subsection, we will examine the Feynman rule $\mathcal{R}_{\text{ule}}^{\text{II}}$ of triple pole by eight examples in figure 10.

Example figure 10.a: this is a six-point example with CHY-integrand

$$\frac{1}{z_{12}^4 z_{34}^2 z_{56}^2 z_{36} z_{45} z_{35} z_{46}}. \quad (4.8)$$

The possible subsets for poles are $\{\underline{1, 2}\}, \{3, 4\}, \{5, 6\}, \{3, 4, 5\}, \{3, 4, 6\}, \{3, 5, 6\}, \{4, 5, 6\}$, so we can construct five compatible combinations as

$$\begin{aligned}
 & \{\{\underline{1, 2}\}, \{3, 4\}, \{5, 6\}\}, \quad \{\{\underline{1, 2}\}, \{3, 4\}, \{3, 4, 5\}\}, \\
 & \{\{\underline{1, 2}\}, \{3, 4\}, \{3, 4, 6\}\}, \quad \{\{\underline{1, 2}\}, \{5, 6\}, \{3, 5, 6\}\}, \quad \{\{\underline{1, 2}\}, \{5, 6\}, \{4, 5, 6\}\}.
 \end{aligned}$$

The five Feynman diagrams then follows in figure 17. By $\mathcal{R}_{\text{ule}}^{\text{II}}$ we get the result as

$$\frac{1}{s_{34}s_{56}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{3, 4\}, \{5, 6\}] + \frac{1}{s_{34}s_{345}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{3, 4, 5\}, \{6\}]$$

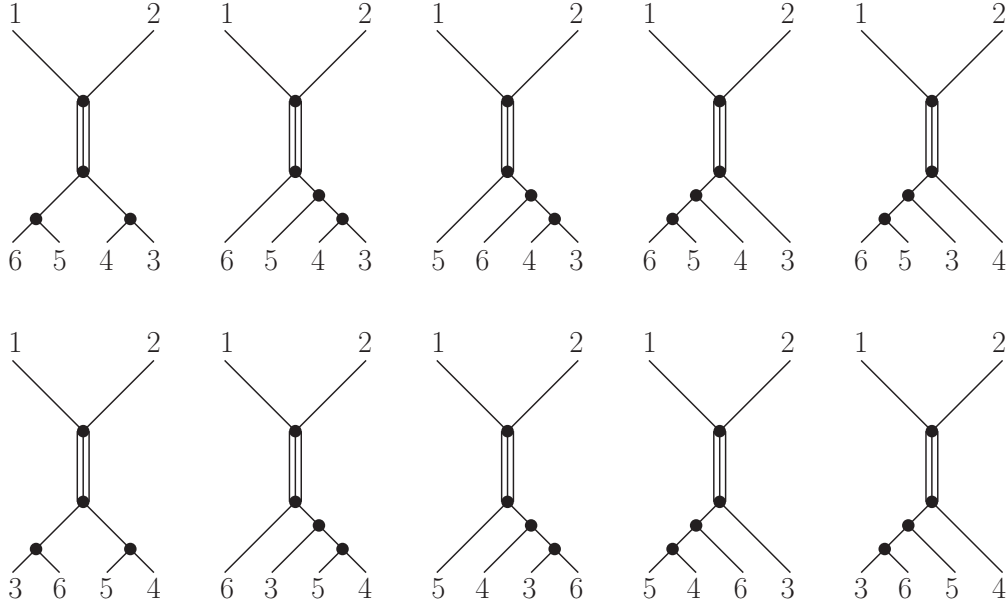


Figure 18. The 10 Feynman diagrams corresponding to the 4-regular graph figure 10.b.

$$\begin{aligned}
 & + \frac{1}{s_{34}s_{346}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{3, 4, 6\}, \{5\}] \\
 & + \frac{1}{s_{56}s_{356}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{3, 5, 6\}, \{4\}] + \frac{1}{s_{56}s_{456}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{4, 5, 6\}, \{3\}] \\
 & = -\frac{s_{15}}{s_{12}^2 s_{34} s_{56}} - \frac{s_{23} s_{15}}{s_{12}^3 s_{34} s_{56}} - \frac{s_{24} s_{15}}{s_{12}^3 s_{34} s_{56}} + \frac{1}{s_{12}^2 s_{56}} - \frac{s_{16}}{s_{12}^2 s_{34} s_{56}} - \frac{s_{16} s_{23}}{s_{12}^3 s_{34} s_{56}} \\
 & \quad - \frac{s_{16} s_{24}}{s_{12}^3 s_{34} s_{56}} + \frac{s_{13} s_{23}}{s_{12}^3 s_{56} s_{456}} + \frac{s_{16} s_{26}}{s_{12}^3 s_{34} s_{345}} + \frac{s_{15} s_{25}}{s_{12}^3 s_{34} s_{346}} + \frac{s_{14} s_{24}}{s_{12}^3 s_{56} s_{356}}. \quad (4.9)
 \end{aligned}$$

Example figure 10.b: this is a six-point example with CHY-integrand

$$\frac{1}{z_{12}^4 z_{34}^2 z_{45}^2 z_{56}^2 z_{16}^2}. \quad (4.10)$$

The possible subsets for poles are $\{\underline{1, 2}\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{3, 6\}, \{3, 4, 5\}, \{3, 4, 6\}, \{3, 5, 6\}, \{4, 5, 6\}$, so the 10 compatible combinations can be given as

$$\begin{aligned}
 & \{\{\underline{1, 2}\}, \{3, 4\}, \{5, 6\}\}, \quad \{\{\underline{1, 2}\}, \{4, 5\}, \{3, 6\}\}, \quad \{\{\underline{1, 2}\}, \{3, 4\}, \{3, 4, 5\}\}, \quad \{\{\underline{1, 2}\}, \{3, 4\}, \{3, 4, 6\}\}, \\
 & \{\{\underline{1, 2}\}, \{5, 6\}, \{3, 5, 6\}\}, \quad \{\{\underline{1, 2}\}, \{5, 6\}, \{4, 5, 6\}\}, \quad \{\{\underline{1, 2}\}, \{4, 5\}, \{3, 4, 5\}\}, \quad \{\{\underline{1, 2}\}, \{3, 6\}, \{3, 4, 6\}\}, \\
 & \{\{\underline{1, 2}\}, \{3, 6\}, \{3, 5, 6\}\}, \quad \{\{\underline{1, 2}\}, \{4, 5\}, \{4, 5, 6\}\},
 \end{aligned}$$

and the corresponding 10 Feynman diagrams are presented in figure 18. Accordingly, we write down the result by $\mathcal{R}_{\text{ule}}^{\text{II}}$ as

$$\begin{aligned}
 & \frac{1}{s_{34}s_{56}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{3, 4\}, \{5, 6\}] + \frac{1}{s_{36}s_{45}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{3, 6\}, \{4, 5\}] \\
 & + \left(\frac{1}{s_{34}s_{345}} + \frac{1}{s_{45}s_{345}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{3, 4, 5\}, \{6\}]
 \end{aligned}$$

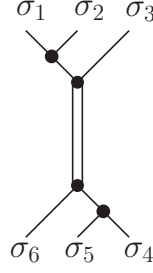


Figure 19. The nine Feynman diagrams corresponding to the 4-regular graph figure 10.c. $\{\sigma_1, \sigma_2, \sigma_3\}$ takes the cyclic permutations of $\{1, 2, 3\}$, and $\{\sigma_4, \sigma_5, \sigma_6\}$ takes the cyclic permutation of $\{4, 5, 6\}$.

$$\begin{aligned}
 & + \left(\frac{1}{s_{34}s_{346}} + \frac{1}{s_{36}s_{346}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{3, 4, 6\}, \{5\}] \\
 & + \left(\frac{1}{s_{56}s_{356}} + \frac{1}{s_{36}s_{356}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{3, 5, 6\}, \{4\}] \\
 & + \left(\frac{1}{s_{56}s_{456}} + \frac{1}{s_{45}s_{456}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{1\}, \{2\}, \{4, 5, 6\}, \{3\}].
 \end{aligned} \tag{4.11}$$

Example figure 10.c: this is a six-point example with CHY-integrand

$$\frac{1}{z_{12}^2 z_{23}^2 z_{31}^2 z_{45}^2 z_{56}^2 z_{64}^2}. \tag{4.12}$$

The possible subsets for poles are $\{\underline{1, 2, 3}\}, \{1, 2\}, \{2, 3\}, \{1, 2\}, \{4, 5\}, \{5, 6\}, \{4, 6\}$, so we can construct nine compatible combinations as

$$\begin{aligned}
 & \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{4, 5\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{5, 6\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{4, 6\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{4, 5\}\}, \quad \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{5, 6\}\}, \quad \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{4, 6\}\} \\
 & \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{4, 5\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{5, 6\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{4, 6\}\},
 \end{aligned}$$

with nine Feynman diagrams as shown in figure 19. Then the answer is trivially given by

$$\begin{aligned}
 & \frac{1}{s_{12}s_{45}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2\}, \{3\}, \{4, 5\}, \{6\}] + \frac{1}{s_{12}s_{56}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2\}, \{3\}, \{5, 6\}, \{4\}] \\
 & + \frac{1}{s_{12}s_{46}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2\}, \{3\}, \{4, 6\}, \{5\}] + \frac{1}{s_{23}s_{45}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{2, 3\}, \{1\}, \{4, 5\}, \{6\}] \\
 & + \frac{1}{s_{23}s_{56}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{2, 3\}, \{1\}, \{5, 6\}, \{4\}] + \frac{1}{s_{23}s_{46}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{2, 3\}, \{1\}, \{4, 6\}, \{5\}] \\
 & + \frac{1}{s_{13}s_{45}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 3\}, \{2\}, \{4, 5\}, \{6\}] + \frac{1}{s_{13}s_{56}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 3\}, \{2\}, \{5, 6\}, \{4\}] \\
 & + \frac{1}{s_{13}s_{46}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 3\}, \{2\}, \{4, 6\}, \{5\}].
 \end{aligned}$$

Example figure 10.d: this is a seven-point example with CHY-integrand

$$- \frac{1}{z_{12}^2 z_{23}^2 z_{31}^2 z_{45}^2 z_{67}^2 z_{56} z_{64} z_{47} z_{75}}. \tag{4.13}$$

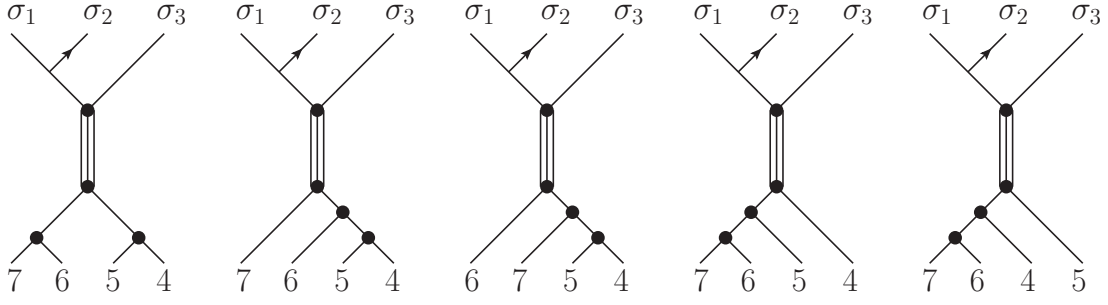


Figure 20. The 15 Feynman diagrams corresponding to the 4-regular graph figure 10.d. $\{\sigma_1, \sigma_2, \sigma_3\}$ takes the cyclic permutations of $\{1, 2, 3\}$.

The possible subsets for poles are $\{\underline{1, 2, 3}\}, \{4, 5, 6\}, \{4, 5, 7\}, \{4, 6, 7\}, \{5, 6, 7\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{4, 5\}, \{6, 7\}$, so we can construct 15 compatible combinations of subsets as

$$\begin{aligned}
 & \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{4, 5\}, \{6, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{4, 5\}, \{4, 5, 6\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{4, 5\}, \{4, 5, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{6, 7\}, \{4, 6, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{6, 7\}, \{5, 6, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{4, 5\}, \{6, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{4, 5\}, \{4, 5, 6\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{4, 5\}, \{4, 5, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{6, 7\}, \{4, 6, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{6, 7\}, \{5, 6, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{4, 5\}, \{4, 5, 6\}\}, \quad \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{4, 5\}, \{4, 5, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{6, 7\}, \{4, 6, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{6, 7\}, \{5, 6, 7\}\}.
 \end{aligned}$$

With them we can draw 15 Feynman diagrams as presented in figure 20. According to rule $\mathcal{R}_{\text{ule}}^{\text{II}}$, the result is given by

$$\begin{aligned}
 & \frac{1}{s_{12}s_{45}s_{67}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2\}, \{3\}, \{4, 5\}, \{6, 7\}] + \frac{1}{s_{12}s_{45}s_{456}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2\}, \{3\}, \{4, 5, 6\}, \{7\}] \\
 & + \frac{1}{s_{12}s_{45}s_{457}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2\}, \{3\}, \{4, 5, 7\}, \{6\}] + \frac{1}{s_{12}s_{67}s_{467}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2\}, \{3\}, \{4, 6, 7\}, \{5\}] \\
 & + \frac{1}{s_{12}s_{67}s_{567}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2\}, \{3\}, \{5, 6, 7\}, \{4\}] + \left(\{1, 2, 3\} \rightarrow \{2, 3, 1\}, \quad \{3, 1, 2\} \right).
 \end{aligned}$$

Example figure 10.e: this is a seven-point example with CHY-integrand

$$- \frac{1}{z_{12}^2 z_{23}^2 z_{31}^2 z_{45}^2 z_{56}^2 z_{67}^2 z_{74}^2}. \quad (4.14)$$

The possible subsets for poles are $\{\underline{1, 2, 3}\}, \{4, 5, 6\}, \{4, 5, 7\}, \{4, 6, 7\}, \{5, 6, 7\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{4, 7\}$, so we can construct 30 compatible combinations as

$$\begin{aligned}
 & \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{4, 5\}, \{6, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{4, 7\}, \{5, 6\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{4, 5\}, \{4, 5, 6\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{4, 5\}, \{4, 5, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{6, 7\}, \{4, 6, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{6, 7\}, \{5, 6, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{5, 6\}, \{4, 5, 6\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{4, 7\}, \{4, 5, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{4, 7\}, \{4, 6, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{1, 2\}, \{5, 6\}, \{5, 6, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{4, 7\}, \{5, 6\}\}, \quad \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{4, 5\}, \{4, 5, 6\}\},
 \end{aligned}$$

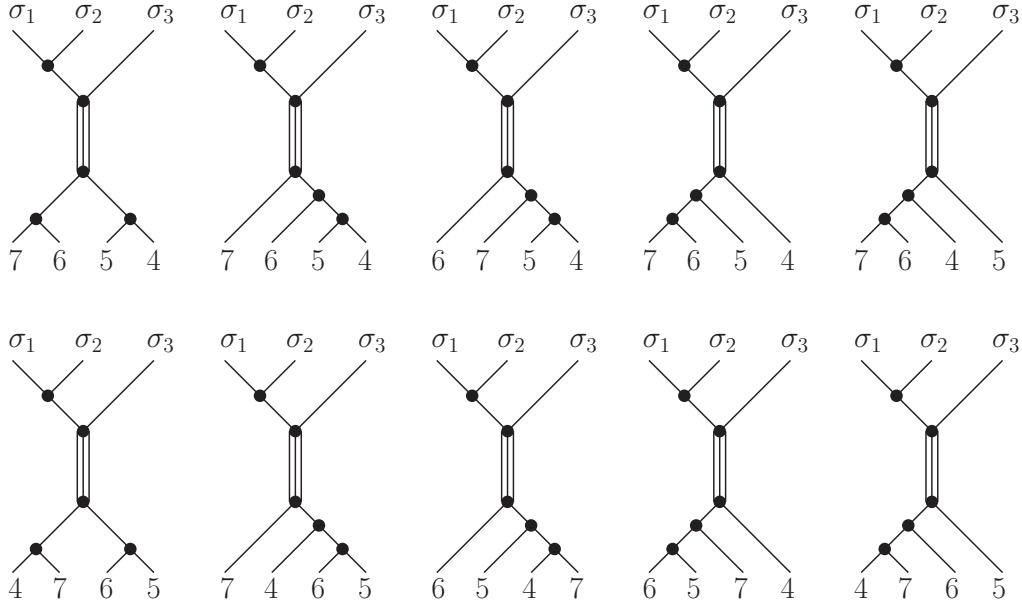


Figure 21. The 30 Feynman diagrams corresponding to the 4-regular graph figure 10.e. $\{\sigma_1, \sigma_2, \sigma_3\}$ takes the cyclic permutations of $\{1, 2, 3\}$.

$$\begin{aligned}
 & \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{4, 5\}, \{4, 5, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{6, 7\}, \{4, 6, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{6, 7\}, \{5, 6, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{5, 6\}, \{4, 5, 6\}\}, \quad \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{4, 7\}, \{4, 5, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{4, 7\}, \{4, 6, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{2, 3\}, \{5, 6\}, \{5, 6, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{4, 5\}, \{6, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{4, 7\}, \{5, 6\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{4, 5\}, \{4, 5, 6\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{4, 5\}, \{4, 5, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{6, 7\}, \{4, 6, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{6, 7\}, \{5, 6, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{5, 6\}, \{4, 5, 6\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{4, 7\}, \{4, 5, 7\}\}, \quad \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{4, 7\}, \{4, 6, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{1, 3\}, \{5, 6\}, \{5, 6, 7\}\}.
 \end{aligned}$$

The 30 Feynman diagrams corresponding to this CHY-integrand is shown in figure 21. Then the answer is given by

$$\begin{aligned}
 & \frac{1}{s_{12}s_{45}s_{67}} \mathcal{R}_{\text{ule}}^{\text{H}}[\{1, 2\}, \{3\}, \{4, 5\}, \{6, 7\}] + \frac{1}{s_{12}s_{47}s_{56}} \mathcal{R}_{\text{ule}}^{\text{H}}[\{1, 2\}, \{3\}, \{4, 7\}, \{5, 6\}] \\
 & + \frac{1}{s_{12}} \left(\frac{1}{s_{45}s_{456}} + \frac{1}{s_{56}s_{456}} \right) \mathcal{R}_{\text{ule}}^{\text{H}}[\{1, 2\}, \{3\}, \{4, 5, 6\}, \{7\}] \\
 & + \frac{1}{s_{12}} \left(\frac{1}{s_{45}s_{457}} + \frac{1}{s_{47}s_{457}} \right) \mathcal{R}_{\text{ule}}^{\text{H}}[\{1, 2\}, \{3\}, \{4, 5, 7\}, \{6\}] \\
 & + \frac{1}{s_{12}} \left(\frac{1}{s_{47}s_{467}} + \frac{1}{s_{67}s_{467}} \right) \mathcal{R}_{\text{ule}}^{\text{H}}[\{1, 2\}, \{3\}, \{4, 6, 7\}, \{5\}] \\
 & + \frac{1}{s_{12}} \left(\frac{1}{s_{56}s_{567}} + \frac{1}{s_{67}s_{567}} \right) \mathcal{R}_{\text{ule}}^{\text{H}}[\{1, 2\}, \{3\}, \{5, 6, 7\}, \{4\}] \\
 & + \left(\{1, 2, 3\} \rightarrow \{2, 3, 1\}, \quad \{3, 1, 2\} \right).
 \end{aligned}$$

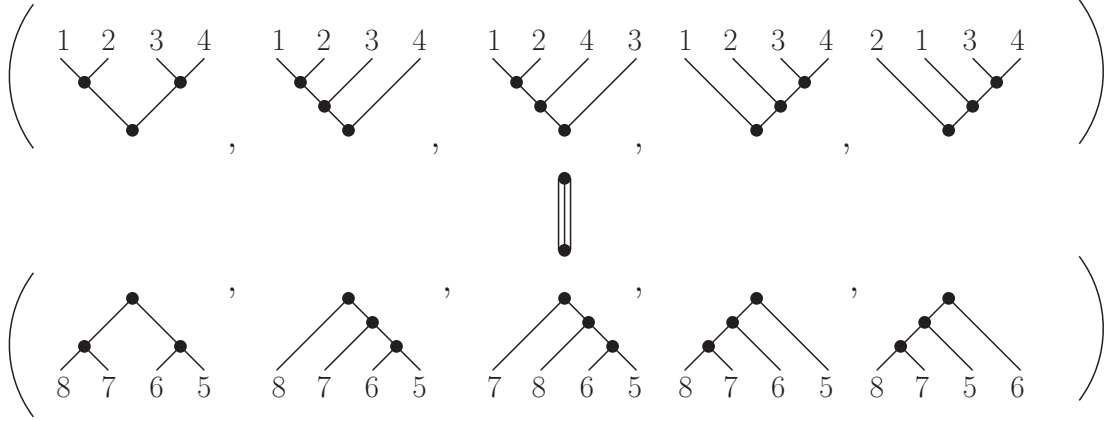


Figure 22. The 25 Feynman diagrams corresponding to the 4-regular graph figure 10.f.

Example figure 10.f: this is an eight-point example with CHY-integrand

$$\frac{1}{z_{12}^2 z_{34}^2 z_{56}^2 z_{78}^2 z_{23} z_{31} z_{14} z_{42} z_{67} z_{75} z_{58} z_{86}}. \quad (4.15)$$

The possible subsets for poles are $\{\underline{1, 2, 3, 4}\}$, $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{1, 3, 4\}$, $\{1, 2, 4\}$, $\{5, 6, 7\}$, $\{6, 7, 8\}$, $\{5, 7, 8\}$, $\{5, 6, 8\}$, $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, $\{7, 8\}$, so we can construct 25 compatible combinations of subsets as

$$\begin{aligned} & \{\{\underline{1, 2, 3, 4}\}, \{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}, \\ & \{\{\underline{1, 2, 3, 4}\}, [\{1, 2, 3\}, \{1, 2\} \text{ or } \{1, 2, 4\}, \{1, 2\} \text{ or } \{1, 3, 4\}, \{3, 4\} \text{ or } \{2, 3, 4\}, \{3, 4\}]\}, \{5, 6\}, \{7, 8\}\}, \\ & \{\{\underline{1, 2, 3, 4}\}, \{1, 2\}, \{3, 4\}, [\{5, 6, 7\}, \{5, 6\} \text{ or } \{5, 6, 8\}, \{5, 6\} \text{ or } \{5, 7, 8\}, \{7, 8\} \text{ or } \{6, 7, 8\}, \{7, 8\}]\}\}, \\ & \{\{\underline{1, 2, 3, 4}\}, [\{1, 2, 3\}, \{1, 2\} \text{ or } \{1, 2, 4\}, \{1, 2\} \text{ or } \{1, 3, 4\}, \{3, 4\} \text{ or } \{2, 3, 4\}, \{3, 4\}]\}, \\ & \quad [\{5, 6, 7\}, \{5, 6\} \text{ or } \{5, 6, 8\}, \{5, 6\} \text{ or } \{5, 7, 8\}, \{7, 8\} \text{ or } \{6, 7, 8\}, \{7, 8\}]\}\}. \end{aligned}$$

The Feynman diagrams for them are presented in figure 22. Then the answer is given by

$$\begin{aligned} & \frac{1}{s_{12}s_{34}s_{56}s_{78}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}] \\ & + \frac{1}{s_{56}s_{78}} \left(\frac{1}{s_{12}s_{123}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2, 3\}, \{4\}, \{5, 6\}, \{7, 8\}] + \frac{1}{s_{12}s_{124}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2, 4\}, \{3\}, \{5, 6\}, \{7, 8\}] \right. \\ & \quad \left. + \frac{1}{s_{34}s_{134}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 3, 4\}, \{2\}, \{5, 6\}, \{7, 8\}] + \frac{1}{s_{34}s_{234}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{2, 3, 4\}, \{1\}, \{5, 6\}, \{7, 8\}] \right) \\ & + \frac{1}{s_{12}s_{34}} \left(\frac{1}{s_{56}s_{567}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2, 3\}, \{4\}, \{5, 6\}, \{7, 8\}] + \frac{1}{s_{12}s_{124}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2, 4\}, \{3\}, \{5, 6\}, \{7, 8\}] \right. \\ & \quad \left. + \frac{1}{s_{34}s_{134}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 3, 4\}, \{2\}, \{5, 6\}, \{7, 8\}] + \frac{1}{s_{34}s_{234}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{2, 3, 4\}, \{1\}, \{5, 6\}, \{7, 8\}] \right) \\ & + \left(\frac{1}{s_{12}s_{123}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2, 3\}, \{4\}, \{\bullet\}, \{\bullet\}] + \frac{1}{s_{12}s_{124}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2, 4\}, \{3\}, \{\bullet\}, \{\bullet\}] \right. \\ & \quad \left. + \frac{1}{s_{34}s_{134}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 3, 4\}, \{2\}, \{\bullet\}, \{\bullet\}] + \frac{1}{s_{34}s_{234}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{2, 3, 4\}, \{1\}, \{\bullet\}, \{\bullet\}] \right) \end{aligned}$$

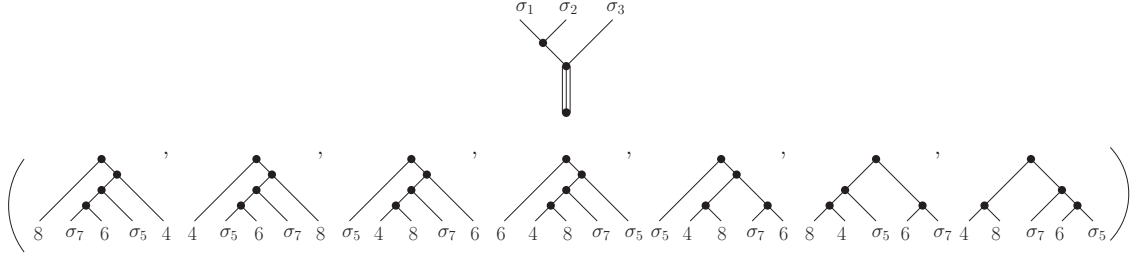


Figure 23. The 42 Feynman diagrams corresponding to the 4-regular graph figure 10.g. $\{\sigma_1, \sigma_2, \sigma_3\}$ takes the cyclic permutations of $\{1, 2, 3\}$, while $\{\sigma_5, \sigma_7\}$ takes the permutations of $\{5, 7\}$.

$$\bigotimes \left(\frac{1}{s_{56}s_{567}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{5, 6, 7\}, \{8\}] + \frac{1}{s_{56}s_{568}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{5, 6, 8\}, \{7\}] \right. \\ \left. + \frac{1}{s_{78}s_{578}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{5, 7, 8\}, \{6\}] + \frac{1}{s_{78}s_{678}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{6, 7, 8\}, \{5\}] \right), \quad (4.16)$$

where in order to simplify the presentation we use product \bigotimes to denote

$$\mathcal{R}_{\text{ule}}^{\text{II}}[P_A, P_B, \bullet, \bullet] \bigotimes \mathcal{R}_{\text{ule}}^{\text{II}}[\bullet, \bullet, P_C, P_D] := \mathcal{R}_{\text{ule}}^{\text{II}}[P_A, P_B, P_C, P_D]. \quad (4.17)$$

Example figure 10.g: this is an eight-point example with CHY-integrand

$$\frac{1}{z_{12}^2 z_{23}^2 z_{31}^2 z_{84}^2 z_{56}^2 z_{67}^2 z_{45} z_{58} z_{87} z_{74}}. \quad (4.18)$$

The possible subsets for poles are $\{\underline{1, 2, 3}\}$, $\{4, 5, 6, 7\}$, $\{5, 6, 7, 8\}$, $\{6, 7, 8, 4\}$, $\{7, 8, 4, 5\}$, $\{8, 4, 5, 6\}$, $\{4, 5, 8\}$, $\{4, 7, 8\}$, $\{5, 6, 7\}$, $\{1, 2\}$, $\{2, 3\}$, $\{1, 3\}$, $\{4, 8\}$, $\{5, 6\}$, $\{6, 7\}$, so we can construct 42 compatible combinations of subsets as

$$\begin{aligned} & \{ \{ \underline{1, 2, 3} \}, \{4, 5, 6, 7\}, \{5, 6, 7\}, [\{5, 6\} \text{ or } \{6, 7\}], [\{1, 2\} \text{ or } \{2, 3\} \text{ or } \{1, 3\}] \}, \\ & \{ \{ \underline{1, 2, 3} \}, \{5, 6, 7, 8\}, \{5, 6, 7\}, [\{5, 6\} \text{ or } \{6, 7\}], [\{1, 2\} \text{ or } \{2, 3\} \text{ or } \{1, 3\}] \}, \\ & \{ \{ \underline{1, 2, 3} \}, \{6, 7, 8, 4\}, [\{6, 7\}, \{4, 8\} \text{ or } \{4, 7, 8\}, \{4, 8\}], [\{1, 2\} \text{ or } \{2, 3\} \text{ or } \{1, 3\}] \}, \\ & \{ \{ \underline{1, 2, 3} \}, \{7, 8, 4, 5\}, [\{4, 5, 8\}, \{4, 8\} \text{ or } \{4, 7, 8\}, \{4, 8\}], [\{1, 2\} \text{ or } \{2, 3\} \text{ or } \{1, 3\}] \}, \\ & \{ \{ \underline{1, 2, 3} \}, \{8, 4, 5, 6\}, [\{4, 5, 8\}, \{4, 8\} \text{ or } \{5, 6\}, \{4, 8\}], [\{1, 2\} \text{ or } \{2, 3\} \text{ or } \{1, 3\}] \}, \\ & \{ \{ \underline{1, 2, 3} \}, \{5, 6, 7\}, \{4, 8\}, [\{5, 6\} \text{ or } \{6, 7\}], [\{1, 2\} \text{ or } \{2, 3\} \text{ or } \{1, 3\}] \}, \\ & \{ \{ \underline{1, 2, 3} \}, \{4, 5, 8\}, \{6, 7\}, \{4, 8\}, [\{1, 2\} \text{ or } \{2, 3\} \text{ or } \{1, 3\}] \}, \\ & \{ \{ \underline{1, 2, 3} \}, \{4, 7, 8\}, \{5, 6\}, \{4, 8\}, [\{1, 2\} \text{ or } \{2, 3\} \text{ or } \{1, 3\}] \}. \end{aligned}$$

The corresponding 42 Feynman diagrams are presented in figure 23. By $\mathcal{R}_{\text{ule}}^{\text{II}}$ we immediately arrive at the result

$$\left(\frac{1}{s_{12}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 2\}, \{3\}, \{\bullet\}, \{\bullet\}] + \frac{1}{s_{23}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{2, 3\}, \{1\}, \{\bullet\}, \{\bullet\}] + \frac{1}{s_{13}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{1, 3\}, \{2\}, \{\bullet\}, \{\bullet\}] \right)$$

$$\begin{aligned}
 & \otimes \left(\frac{1}{s_{4567}s_{567}} \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{4, 5, 6, 7\}, \{8\}] \right. \\
 & + \frac{1}{s_{5678}s_{567}} \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{5, 6, 7, 8\}, \{4\}] \\
 & + \frac{1}{s_{6784}s_{48}} \left(\frac{1}{s_{478}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{6, 7, 8, 4\}, \{5\}] \\
 & + \frac{1}{s_{7845}s_{48}} \left(\frac{1}{s_{458}} + \frac{1}{s_{478}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{7, 8, 4, 5\}, \{6\}] \\
 & + \frac{1}{s_{8456}s_{48}} \left(\frac{1}{s_{458}} + \frac{1}{s_{56}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{8, 4, 5, 6\}, \{7\}] \\
 & + \frac{1}{s_{567}s_{48}} \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{5, 6, 7\}, \{4, 8\}] \\
 & + \frac{1}{s_{458}s_{67}s_{48}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{4, 5, 8\}, \{6, 7\}] \\
 & \left. + \frac{1}{s_{478}s_{56}s_{48}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{4, 7, 8\}, \{5, 6\}] \right). \tag{4.19}
 \end{aligned}$$

Example figure 10.h: this is a nine-point example with CHY-integrand

$$- \frac{1}{z_{12}^2 z_{39}^2 z_{84}^2 z_{56}^2 z_{67}^2 z_{28} z_{91} z_{13} z_{32} z_{47} z_{78} z_{85} z_{54}}. \tag{4.20}$$

The possible subsets for poles are $\{4, 5, 6, 7, 8\} = \{1, 2, 3, 9\}$, $\{5, 6, 7, 8\}$, $\{6, 7, 8, 4\}$, $\{7, 8, 4, 5\}$, $\{8, 4, 5, 6\}$, $\{4, 5, 6, 7\}$, $\{5, 6, 7\}$, $\{7, 8, 4\}$, $\{8, 4, 5\}$, $\{3, 9, 1\}$, $\{3, 9, 2\}$, $\{3, 1, 2\}$, $\{9, 1, 2\}$, $\{5, 6\}$, $\{6, 7\}$, $\{8, 4\}$, $\{3, 9\}$, $\{1, 2\}$, so we can construct 70 compatible combinations of subsets as

$$\begin{aligned}
 & \{ \{ \underline{1, 2, 3, 9} \}, \{5, 6, 7\}, \{5, 6\}, \{8, 4\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \text{ or } \{1, 2\}, \{3, 1, 2\} \\
 & \text{or } \{1, 2\}, \{9, 1, 2\}] \}, \\
 & \{ \{ \underline{1, 2, 3, 9} \}, \{5, 6, 7\}, \{6, 7\}, \{8, 4\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \text{ or } \{1, 2\}, \{3, 1, 2\} \\
 & \text{or } \{1, 2\}, \{9, 1, 2\}] \}, \\
 & \{ \{ \underline{1, 2, 3, 9} \}, \{7, 8, 4\}, \{8, 4\}, \{5, 6\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \text{ or } \{1, 2\}, \{3, 1, 2\} \\
 & \text{or } \{1, 2\}, \{9, 1, 2\}] \}, \\
 & \{ \{ \underline{1, 2, 3, 9} \}, \{8, 4, 5\}, \{8, 4\}, \{6, 7\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \text{ or } \{1, 2\}, \{3, 1, 2\} \\
 & \text{or } \{1, 2\}, \{9, 1, 2\}] \}, \\
 & \{ \{ \underline{1, 2, 3, 9} \}, \{5, 6, 7, 8\}, \{5, 6, 7\}, \{5, 6\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \\
 & \text{or } \{1, 2\}, \{3, 1, 2\} \text{ or } \{1, 2\}, \{9, 1, 2\}] \}, \\
 & \{ \{ \underline{1, 2, 3, 9} \}, \{5, 6, 7, 8\}, \{5, 6, 7\}, \{6, 7\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \\
 & \text{or } \{1, 2\}, \{3, 1, 2\} \text{ or } \{1, 2\}, \{9, 1, 2\}] \}, \\
 & \{ \{ \underline{1, 2, 3, 9} \}, \{6, 7, 8, 4\}, \{6, 7\}, \{8, 4\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \\
 & \text{or } \{1, 2\}, \{3, 1, 2\} \text{ or } \{1, 2\}, \{9, 1, 2\}] \},
 \end{aligned}$$

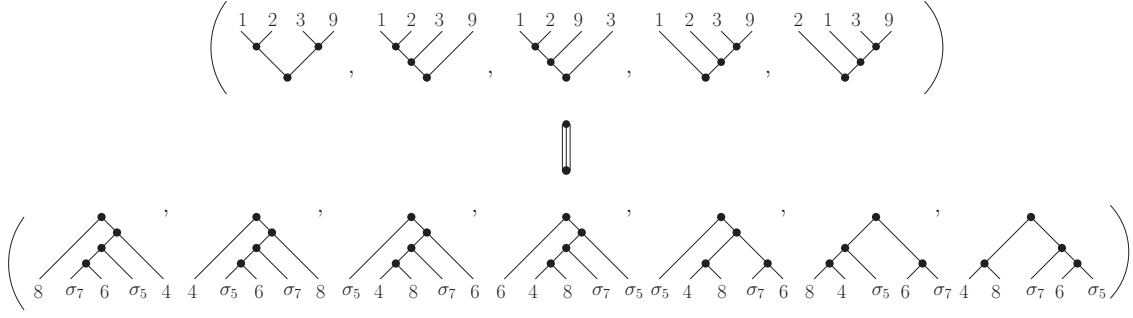


Figure 24. The 70 Feynman diagrams corresponding to the 4-regular graph figure 10.h. $\{\sigma_5, \sigma_7\}$ takes the permutations of $\{5, 7\}$.

$$\begin{aligned}
 & \{ \{ \underline{1, 2, 3, 9} \}, \{6, 7, 8, 4\}, \{7, 8, 4\}, \{8, 4\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \\
 & \text{or } \{1, 2\}, \{3, 1, 2\} \text{ or } \{1, 2\}, \{9, 1, 2\}] \} , \\
 & \{ \{ \underline{1, 2, 3, 9} \}, \{7, 8, 4, 5\}, \{7, 8, 4\}, \{8, 4\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \\
 & \text{or } \{1, 2\}, \{3, 1, 2\} \text{ or } \{1, 2\}, \{9, 1, 2\}] \} , \\
 & \{ \{ \underline{1, 2, 3, 9} \}, \{7, 8, 4, 5\}, \{8, 4, 5\}, \{8, 4\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \\
 & \text{or } \{1, 2\}, \{3, 1, 2\} \text{ or } \{1, 2\}, \{9, 1, 2\}] \} , \\
 & \{ \{ \underline{1, 2, 3, 9} \}, \{8, 4, 5, 6\}, \{8, 4, 5\}, \{8, 4\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \\
 & \text{or } \{1, 2\}, \{3, 1, 2\} \text{ or } \{1, 2\}, \{9, 1, 2\}] \} , \\
 & \{ \{ \underline{1, 2, 3, 9} \}, \{8, 4, 5, 6\}, \{8, 4\}, \{5, 6\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \\
 & \text{or } \{1, 2\}, \{3, 1, 2\} \text{ or } \{1, 2\}, \{9, 1, 2\}] \} , \\
 & \{ \{ \underline{1, 2, 3, 9} \}, \{4, 5, 6, 7\}, \{5, 6, 7\}, \{5, 6\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \\
 & \text{or } \{1, 2\}, \{3, 1, 2\} \text{ or } \{1, 2\}, \{9, 1, 2\}] \} , \\
 & \{ \{ \underline{1, 2, 3, 9} \}, \{4, 5, 6, 7\}, \{5, 6, 7\}, \{6, 7\}, [\{3, 9\}, \{1, 2\} \text{ or } \{3, 9\}, \{3, 9, 1\} \text{ or } \{3, 9\}, \{3, 9, 2\} \\
 & \text{or } \{1, 2\}, \{3, 1, 2\} \text{ or } \{1, 2\}, \{9, 1, 2\}] \} .
 \end{aligned}$$

The corresponding 70 Feynman diagrams for them are shown in figure 24. According to the rule $\mathcal{R}_{\text{ule}}^{\text{II}}$, the result is given by

$$\begin{aligned}
 & \left(\frac{1}{s_{567}s_{84}} \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{5, 6, 7\}, \{8, 4\}, \{\bullet\}, \{\bullet\}] \right. \\
 & + \frac{1}{s_{784}s_{56}s_{84}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{5, 6\}, \{7, 8, 4\}, \{\bullet\}, \{\bullet\}] + \frac{1}{s_{845}s_{84}s_{67}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{8, 4, 5\}, \{6, 7\}, \{\bullet\}, \{\bullet\}] \\
 & + \frac{1}{s_{5678}s_{567}} \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{5, 6, 7, 8\}, \{4\}, \{\bullet\}, \{\bullet\}] \\
 & + \frac{1}{s_{6784}s_{84}} \left(\frac{1}{s_{67}} + \frac{1}{s_{784}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{6, 7, 8, 4\}, \{5\}, \{\bullet\}, \{\bullet\}] \\
 & \left. + \frac{1}{s_{7845}s_{84}} \left(\frac{1}{s_{784}} + \frac{1}{s_{845}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{7, 8, 4, 5\}, \{6\}, \{\bullet\}, \{\bullet\}] \right)
 \end{aligned}$$

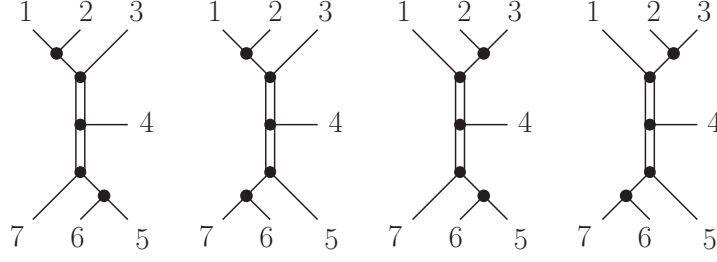


Figure 25. The four Feynman diagrams corresponding to the *4-regular* graph figure 11.a.

$$\begin{aligned}
 & + \frac{1}{s_{8456}s_{84}} \left(\frac{1}{s_{845}} + \frac{1}{s_{56}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{8, 4, 5, 6\}, \{7\}, \{\bullet\}, \{\bullet\}] \\
 & + \frac{1}{s_{4567}s_{567}} \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{II}}[\{4, 5, 6, 7\}, \{8\}, \{\bullet\}, \{\bullet\}] \\
 & \otimes \left(\frac{1}{s_{39}s_{12}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{3, 9\}, \{1, 2\}] + \frac{1}{s_{39}s_{391}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{3, 9, 1\}, \{2\}] \right. \\
 & \quad + \frac{1}{s_{39}s_{392}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{3, 9, 2\}, \{1\}] \\
 & \quad \left. + \frac{1}{s_{12}s_{312}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{3, 1, 2\}, \{9\}] + \frac{1}{s_{12}s_{912}} \mathcal{R}_{\text{ule}}^{\text{II}}[\{\bullet\}, \{\bullet\}, \{9, 1, 2\}, \{3\}] \right). \quad (4.21)
 \end{aligned}$$

4.3 Examples with only duplex-double poles and simple poles

In this subsection, we examine Feynman rule $\mathcal{R}_{\text{ule}}^{\text{III}}$ of duplex-double pole by four examples in figure 11.

Example figure 11.a: this is a seven-point example with CHY-integrand

$$- \frac{1}{z_{12}^2 z_{23}^2 z_{56}^2 z_{67}^2 z_{34} z_{45} z_{57} z_{74} z_{41} z_{13}}. \quad (4.22)$$

The possible subsets for poles are $\{\underline{1, 2, 3}\}$, $\{\underline{5, 6, 7}\}$, $\{1, 2\}$, $\{2, 3\}$, $\{5, 6\}$, $\{6, 7\}$, so we can construct four compatible combinations of subsets as

$$\begin{aligned}
 & \{\{\underline{1, 2, 3}\}, \{\underline{5, 6, 7}\}, \{1, 2\}, \{5, 6\}\}, & \{\{\underline{1, 2, 3}\}, \{\underline{5, 6, 7}\}, \{1, 2\}, \{6, 7\}\}, \\
 & \{\{\underline{1, 2, 3}\}, \{\underline{5, 6, 7}\}, \{2, 3\}, \{5, 6\}\}, & \{\{\underline{1, 2, 3}\}, \{\underline{5, 6, 7}\}, \{2, 3\}, \{6, 7\}\}.
 \end{aligned}$$

From them we can draw four Feynman diagrams as shown in figure 25. Immediately we obtain the result by using $\mathcal{R}_{\text{ule}}^{\text{III}}$,

$$\begin{aligned}
 & \frac{1}{s_{12}s_{56}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1, 2\}, \{3\}, \{4\}, \{5, 6\}, \{7\}] + \frac{1}{s_{12}s_{67}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1, 2\}, \{3\}, \{4\}, \{5\}, \{6, 7\}] \quad (4.23) \\
 & + \frac{1}{s_{23}s_{56}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2, 3\}, \{4\}, \{5, 6\}, \{7\}] + \frac{1}{s_{23}s_{67}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2, 3\}, \{4\}, \{5\}, \{6, 7\}].
 \end{aligned}$$

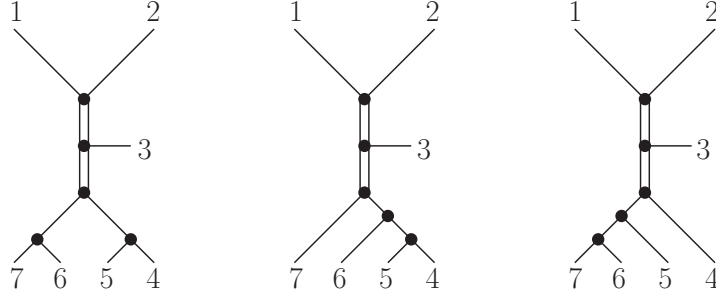


Figure 26. The three Feynman diagrams corresponding to the 4-regular graph figure 11.b.

Example figure 11.b: this is a seven-point example with CHY-integrand

$$\frac{1}{z_{12}^3 z_{45}^2 z_{67}^2 z_{23} z_{34} z_{46} z_{65} z_{57} z_{73} z_{31}}. \quad (4.24)$$

The possible subsets for poles are $\{1, 2\}$, $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{5, 6, 7\}$, $\{4, 5\}$, $\{6, 7\}$, so we can construct three compatible combinations of subsets as

$$\begin{aligned} & \{\{1, 2\}, \{1, 2, 3\}, \{4, 5\}, \{6, 7\}\}, \quad \{\{1, 2\}, \{1, 2, 3\}, \{4, 5\}, \{4, 5, 6\}\}, \\ & \{\{1, 2\}, \{1, 2, 3\}, \{6, 7\}, \{5, 6, 7\}\}. \end{aligned}$$

The corresponding three Feynman diagrams are shown in figure 26. So we can write the answer as

$$\begin{aligned} & \frac{1}{s_{45}s_{67}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2\}, \{3\}, \{4, 5\}, \{6, 7\}] + \frac{1}{s_{45}s_{456}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2\}, \{3\}, \{4, 5, 6\}, \{7\}] \\ & + \frac{1}{s_{67}s_{567}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2\}, \{3\}, \{4\}, \{5, 6, 7\}]. \end{aligned} \quad (4.25)$$

Example figure 11.c: this is an eight-point example with CHY-integrand

$$\frac{1}{z_{12}^2 z_{23}^2 z_{56}^2 z_{67}^2 z_{78}^2 z_{13} z_{14} z_{34} z_{45} z_{48} z_{58}}. \quad (4.26)$$

The possible subsets for poles are $\{1, 2, 3\}$, $\{5, 6, 7, 8\}$, $\{5, 6, 7\}$, $\{6, 7, 8\}$, $\{1, 2\}$, $\{2, 3\}$, $\{5, 6\}$, $\{6, 7\}$, $\{7, 8\}$, so we can construct 10 compatible combinations of subsets as

$$\begin{aligned} & \{\{1, 2, 3\}, \{5, 6, 7, 8\}, [\{1, 2\} \text{ or } \{2, 3\}], \{5, 6, 7\}, [\{5, 6\} \text{ or } \{6, 7\}]\}, \\ & \{\{1, 2, 3\}, \{5, 6, 7, 8\}, [\{1, 2\} \text{ or } \{2, 3\}], \{6, 7, 8\}, [\{6, 7\} \text{ or } \{7, 8\}]\}, \\ & \{\{1, 2, 3\}, \{5, 6, 7, 8\}, [\{1, 2\} \text{ or } \{2, 3\}], \{5, 6\}, \{7, 8\}\}. \end{aligned}$$

The 10 Feynman diagrams can be accordingly drawn as in figure 27. So according to the rule $\mathcal{R}_{\text{ule}}^{\text{III}}$, we can write down the answer as

$$\frac{1}{s_{12}s_{567}} \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{III}}[\{1, 2\}, \{3\}, \{4\}, \{5, 6, 7\}, \{8\}]$$

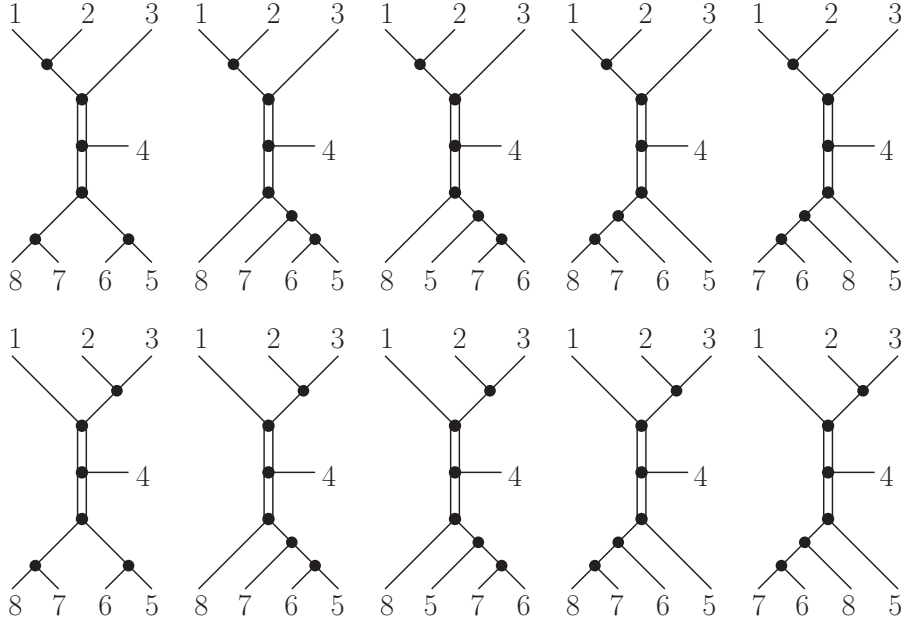


Figure 27. The 10 Feynman diagrams corresponding to the 4-regular graph figure 11.c.

$$\begin{aligned}
 & + \frac{1}{s_{12}s_{678}} \left(\frac{1}{s_{67}} + \frac{1}{s_{78}} \right) \mathcal{R}_{\text{ule}}^{\text{III}}[\{1, 2\}, \{3\}, \{4\}, \{5\}, \{6, 7, 8\}] \\
 & + \frac{1}{s_{12}s_{56}s_{78}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1, 2\}, \{3\}, \{4\}, \{5, 6\}, \{7, 8\}] \\
 & + \frac{1}{s_{23}s_{567}} \left(\frac{1}{s_{56}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2, 3\}, \{4\}, \{5, 6, 7\}, \{8\}] \\
 & + \frac{1}{s_{23}s_{678}} \left(\frac{1}{s_{67}} + \frac{1}{s_{78}} \right) \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2, 3\}, \{4\}, \{5\}, \{6, 7, 8\}] \\
 & + \frac{1}{s_{23}s_{56}s_{78}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2, 3\}, \{4\}, \{5, 6\}, \{7, 8\}].
 \end{aligned} \tag{4.27}$$

Example figure 11.d: this is a nine-point example with CHY-integrand

$$\frac{1}{z_{12}^2 z_{34}^2 z_{67}^2 z_{89}^2 z_{23} z_{31} z_{15} z_{56} z_{68} z_{87} z_{79} z_{95} z_{54} z_{42}}. \tag{4.28}$$

The possible groups for poles are $\{1, 2, 3, 4\}$, $\{6, 7, 8, 9\}$, $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{6, 7, 8\}$, $\{7, 8, 9\}$, $\{1, 2\}$, $\{3, 4\}$, $\{6, 7\}$, $\{8, 9\}$, so we can construct nine compatible combinations of subsets as

$$\begin{aligned}
 & \{ \{1, 2, 3, 4\}, \{6, 7, 8, 9\}, \{1, 2\}, \{3, 4\}, [\{6, 7\}, \{8, 9\} \text{ or } \{6, 7\}, \{6, 7, 8\} \text{ or } \{8, 9\}, \{7, 8, 9\}] \}, \\
 & \{ \{1, 2, 3, 4\}, \{6, 7, 8, 9\}, \{1, 2, 3\}, \{1, 2\}, [\{6, 7\}, \{8, 9\} \text{ or } \{6, 7\}, \{6, 7, 8\} \text{ or } \{8, 9\}, \{7, 8, 9\}] \}, \\
 & \{ \{1, 2, 3, 4\}, \{6, 7, 8, 9\}, \{2, 3, 4\}, \{3, 4\}, [\{6, 7\}, \{8, 9\} \text{ or } \{6, 7\}, \{6, 7, 8\} \text{ or } \{8, 9\}, \{7, 8, 9\}] \}.
 \end{aligned}$$

From them we can draw nine Feynman diagrams as in figure 28. According to the rule

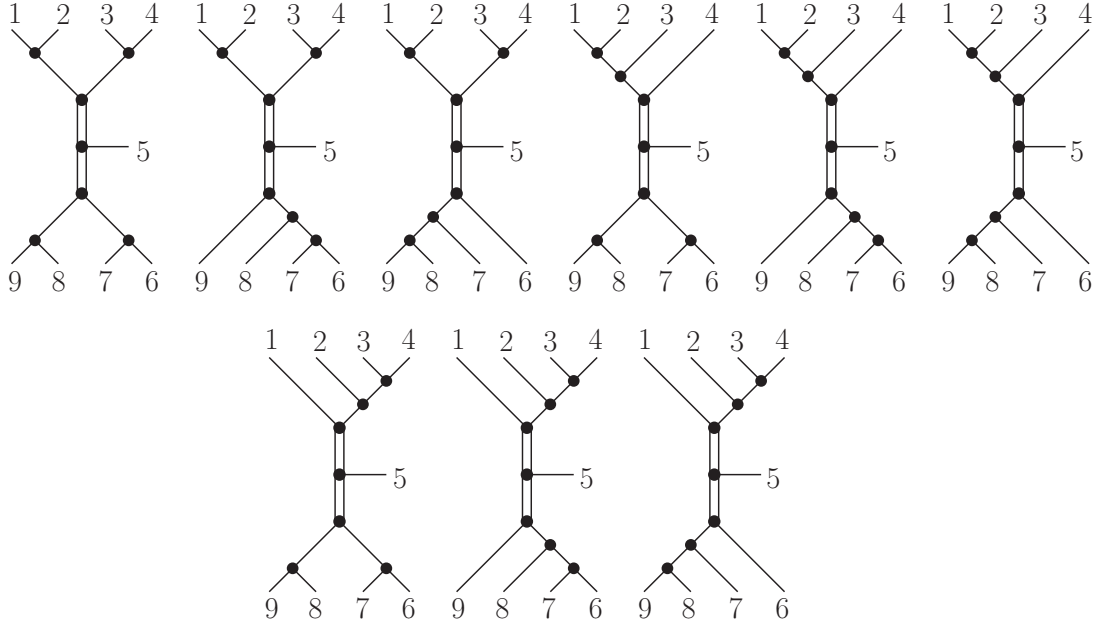


Figure 28. The nine Feynman diagrams corresponding to the 4-regular graph figure 11.d.

$\mathcal{R}_{\text{ule}}^{\text{III}}$, the result is given by

$$\begin{aligned}
 & \frac{1}{s_{12}s_{34}s_{67}s_{89}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1, 2\}, \{3, 4\}, \{5\}, \{6, 7\}, \{8, 9\}] \\
 & + \frac{1}{s_{12}s_{34}s_{67}s_{678}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1, 2\}, \{3, 4\}, \{5\}, \{6, 7, 8\}, \{9\}] \\
 & + \frac{1}{s_{12}s_{34}s_{89}s_{789}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1, 2\}, \{3, 4\}, \{5\}, \{6\}, \{7, 8, 9\}] \\
 & + \frac{1}{s_{12}s_{123}s_{67}s_{89}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1, 2, 3\}, \{4\}, \{5\}, \{6, 7\}, \{8, 9\}] \\
 & + \frac{1}{s_{12}s_{123}s_{67}s_{678}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1, 2, 3\}, \{4\}, \{5\}, \{6, 7, 8\}, \{9\}] \\
 & + \frac{1}{s_{12}s_{123}s_{89}s_{789}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1, 2, 3\}, \{4\}, \{5\}, \{6\}, \{7, 8, 9\}] \\
 & + \frac{1}{s_{34}s_{234}s_{67}s_{89}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2, 3, 4\}, \{5\}, \{6, 7\}, \{8, 9\}] \\
 & + \frac{1}{s_{34}s_{234}s_{67}s_{678}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2, 3, 4\}, \{5\}, \{6, 7, 8\}, \{9\}] \\
 & + \frac{1}{s_{34}s_{234}s_{89}s_{789}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2, 3, 4\}, \{5\}, \{6\}, \{7, 8, 9\}]. \tag{4.29}
 \end{aligned}$$

4.4 Examples with mixed types of higher-order poles

In this subsection, we will examine Feynman rules for higher-order poles by examples in figure 12. They all contain Feynman diagrams with mixed types of higher-order poles, thus really serve as extremely non-trivial supporting for the validation of rules.

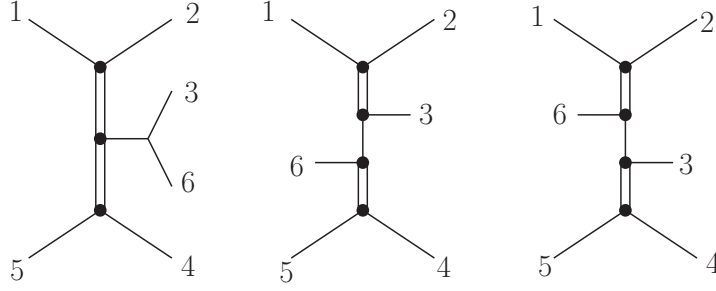


Figure 29. The three Feynman diagrams corresponding to the 4 -regular graph figure 12.a.

Example figure 12.a: this is a six-point example with CHY-integrand

$$\frac{1}{z_{12}^3 z_{45}^3 z_{63}^2 z_{23} z_{34} z_{56} z_{61}}. \quad (4.30)$$

It is simple to find the possible subsets for poles as $\{\underline{1, 2}\}$, $\{\underline{4, 5}\}$, $\{1, 2, 3\}$, $\{1, 2, 6\}$, $\{3, 6\}$. So we can construct three compatible combinations of subsets as

$$\{\{\underline{1, 2}\}, \{\underline{4, 5}\}, \{3, 6\}\}, \quad \{\{\underline{1, 2}\}, \{\underline{4, 5}\}, \{1, 2, 3\}\}, \quad \{\{\underline{1, 2}\}, \{\underline{4, 5}\}, \{1, 2, 6\}\}.$$

From them we draw three Feynman diagrams as presented in figure 29. The first Feynman diagram contains a duplex-double pole while the remaining two contain two double poles. Remind that for the rule $\mathcal{R}_{\text{ule}}^{\text{III}}$ defined in (3.18), we mentioned that the terms with P_E^2 can not be inferred from result of five-point CHY-integrand since therein $P_E^2 = 0$. But it is crucial for generic CHY-integrands whose Feynman diagrams could have massive P_E . This example is indeed the one that motives us to think about adding P_E^2 terms in the Feynman rule.

Let us apply rules $\mathcal{R}_{\text{ule}}^{\text{I}}$ and $\mathcal{R}_{\text{ule}}^{\text{III}}$ for corresponding propagators in Feynman diagrams, which gives

$$\begin{aligned} & \frac{1}{s_{36}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2\}, \{3, 6\}, \{4\}, \{5\}] \\ & + \frac{1}{s_{123}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2\}, \{3\}, \{4, 5, 6\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{6\}, \{1, 2, 3\}, \{4\}, \{5\}] \\ & + \frac{1}{s_{126}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2\}, \{3, 4, 5\}, \{6\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{6, 1, 2\}, \{3\}, \{4\}, \{5\}] \\ & = \frac{s_{15}s_{24} - s_{14}s_{25}}{s_{12}^2 s_{45}^2 s_{36}} + \frac{s_{13}s_{46}}{s_{12}^2 s_{45}^2 s_{123}} + \frac{s_{26}s_{35}}{s_{12}^2 s_{45}^2 s_{126}} - \frac{s_{135}}{s_{12}^2 s_{45}^2}. \end{aligned} \quad (4.31)$$

Example figure 12.b: this is a seven-point example with CHY-integrand

$$-\frac{1}{z_{12}^3 z_{45}^3 z_{67}^2 z_{73}^2 z_{23} z_{34} z_{56} z_{61}}. \quad (4.32)$$

The possible subsets for poles are $\{\underline{1, 2}\}$, $\{\underline{4, 5}\}$, $\{1, 2, 3\}$, $\{1, 2, 6\}$, $\{3, 4, 5\}$, $\{4, 5, 6\}$, $\{3, 6, 7\}$, $\{3, 7\}$, $\{6, 7\}$. From them we can construct eight compatible combinations of subsets as

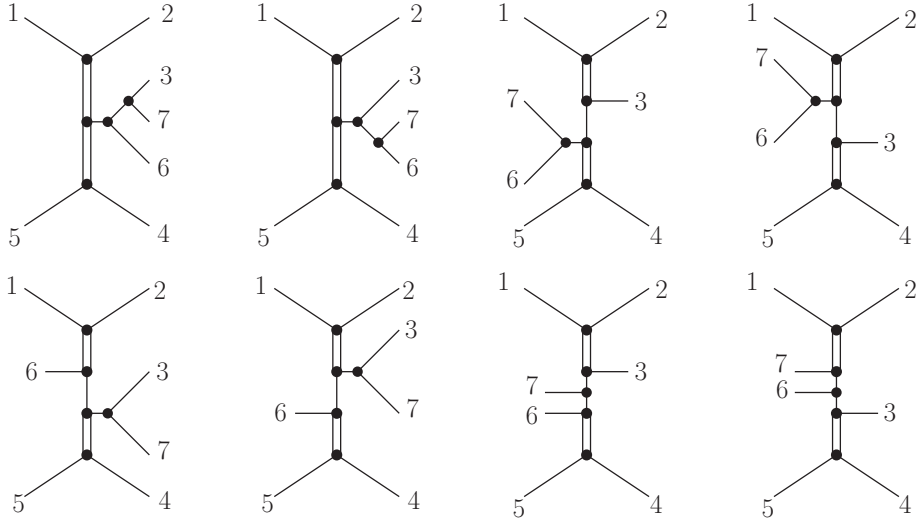


Figure 30. The eight Feynman diagrams corresponding to the *4-regular* graph figure 12.b.

$$\begin{aligned}
 & \{\{1, 2\}, \{4, 5\}, \{3, 6, 7\}, \{3, 7\}\}, & \{\{1, 2\}, \{4, 5\}, \{3, 6, 7\}, \{6, 7\}\}, & \{\{1, 2\}, \{4, 5\}, \{1, 2, 3\}, \{6, 7\}\}, \\
 & \{\{1, 2\}, \{4, 5\}, \{3, 4, 5\}, \{6, 7\}\}, & \{\{1, 2\}, \{4, 5\}, \{1, 2, 6\}, \{3, 7\}\}, & \{\{1, 2\}, \{4, 5\}, \{4, 5, 6\}, \{3, 7\}\}, \\
 & \{\{1, 2\}, \{4, 5\}, \{1, 2, 3\}, \{4, 5, 6\}\}, & \{\{1, 2\}, \{4, 5\}, \{1, 2, 6\}, \{3, 4, 5\}\}.
 \end{aligned}$$

The corresponding eight Feynman diagrams are shown in figure 30. So applying the rules we get the result

$$\begin{aligned}
 & \frac{1}{s_{367}} \left(\frac{1}{s_{37}} + \frac{1}{s_{67}} \right) \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2\}, \{3, 6, 7\}, \{4\}, \{5\}] \\
 & + \frac{1}{s_{123}s_{67}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2\}, \{3\}, \{4, 5, 6, 7\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{6, 7\}, \{1, 2, 3\}, \{4\}, \{5\}] \\
 & + \frac{1}{s_{345}s_{67}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2\}, \{3, 4, 5\}, \{6, 7\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{6, 7, 1, 2\}, \{3\}, \{4\}, \{5\}] \\
 & + \frac{1}{s_{126}s_{37}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2\}, \{3, 4, 5, 7\}, \{6\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{6, 1, 2\}, \{3, 7\}, \{4\}, \{5\}] \\
 & + \frac{1}{s_{456}s_{37}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2\}, \{3, 7\}, \{4, 5, 6\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{6\}, \{1, 2, 3, 7\}, \{4\}, \{5\}] \\
 & + \frac{1}{s_{123}s_{456}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2\}, \{3\}, \{4, 5, 6, 7\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{6\}, \{7, 1, 2, 3\}, \{4\}, \{5\}] \\
 & + \frac{1}{s_{126}s_{345}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2\}, \{7, 3, 4, 5\}, \{6\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{6, 7, 1, 2\}, \{3\}, \{4\}, \{5\}]. \quad (4.33)
 \end{aligned}$$

Example figure 12.c: this is an eight-point example with CHY-integrand

$$\frac{1}{z_{12}^2 z_{34}^2 z_{85}^2 z_{67}^3 z_{23} z_{31} z_{18} z_{87} z_{65} z_{54} z_{42}}. \quad (4.34)$$

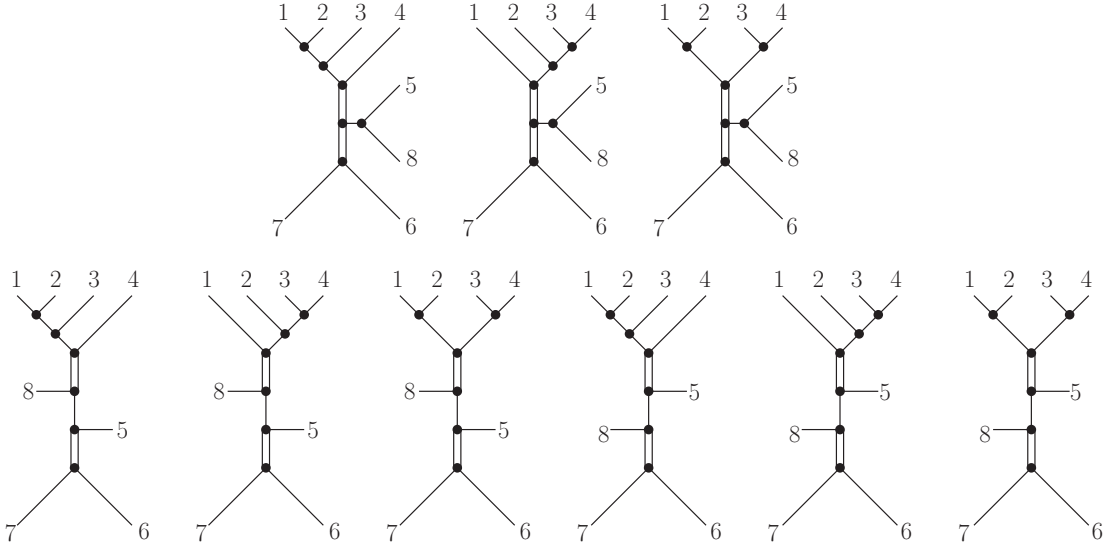


Figure 31. The nine Feynman diagrams corresponding to the 4-regular graph figure 12.c.

The possible subsets for poles are $\{1, 2, 3, 4\}$, $\{6, 7\}$, $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{5, 6, 7\}$, $\{6, 7, 8\}$, $\{1, 2\}$, $\{3, 4\}$, $\{5, 8\}$. So we can construct nine compatible combinations of subsets as

$$\begin{aligned}
 & \{\{1, 2, 3, 4\}, \{6, 7\}, \{1, 2, 3\}, \{1, 2\}, \quad \left[\{5, 8\} \text{ or } \{5, 6, 7\} \text{ or } \{6, 7, 8\} \right], \\
 & \{\{1, 2, 3, 4\}, \{6, 7\}, \{2, 3, 4\}, \{3, 4\}, \quad \left[\{5, 8\} \text{ or } \{5, 6, 7\} \text{ or } \{6, 7, 8\} \right], \\
 & \{\{1, 2, 3, 4\}, \{6, 7\}, \{1, 2\}, \{3, 4\}, \quad \left[\{5, 8\} \text{ or } \{5, 6, 7\} \text{ or } \{6, 7, 8\} \right].
 \end{aligned}$$

From them we can draw nine Feynman diagrams as shown in figure 31. So according to the rules, the result is given by

$$\begin{aligned}
 & \frac{1}{s_{123}s_{12}s_{58}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1, 2, 3\}, \{4\}, \{5, 8\}, \{6\}, \{7\}] + \frac{1}{s_{234}s_{34}s_{58}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1\}, \{2, 3, 4\}, \{5, 8\}, \{6\}, \{7\}] \\
 & + \frac{1}{s_{12}s_{34}s_{58}} \mathcal{R}_{\text{ule}}^{\text{III}}[\{1, 2\}, \{3, 4\}, \{5, 8\}, \{6\}, \{7\}] \\
 & + \frac{1}{s_{123}s_{12}s_{567}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1, 2, 3\}, \{4\}, \{5, 6, 7\}, \{8\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{8, 1, 2, 3, 4\}, \{5\}, \{6\}, \{7\}] \\
 & + \frac{1}{s_{234}s_{34}s_{567}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2, 3, 4\}, \{5, 6, 7\}, \{8\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{8, 1, 2, 3, 4\}, \{5\}, \{6\}, \{7\}] \\
 & + \frac{1}{s_{12}s_{34}s_{567}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1, 2\}, \{3, 4\}, \{5, 6, 7\}, \{8\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{8, 1, 2, 3, 4\}, \{5\}, \{6\}, \{7\}] \\
 & + \frac{1}{s_{123}s_{12}s_{678}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1, 2, 3\}, \{4\}, \{5\}, \{6, 7, 8\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{8\}, \{1, 2, 3, 4, 5\}, \{6\}, \{7\}] \\
 & + \frac{1}{s_{234}s_{34}s_{678}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1\}, \{2, 3, 4\}, \{5\}, \{6, 7, 8\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{8\}, \{1, 2, 3, 4, 5\}, \{6\}, \{7\}] \\
 & + \frac{1}{s_{12}s_{34}s_{678}} \mathcal{R}_{\text{ule}}^{\text{I}}[\{1, 2\}, \{3, 4\}, \{5\}, \{6, 7, 8\}] \mathcal{R}_{\text{ule}}^{\text{I}}[\{8\}, \{1, 2, 3, 4, 5\}, \{6\}, \{7\}]. \tag{4.35}
 \end{aligned}$$

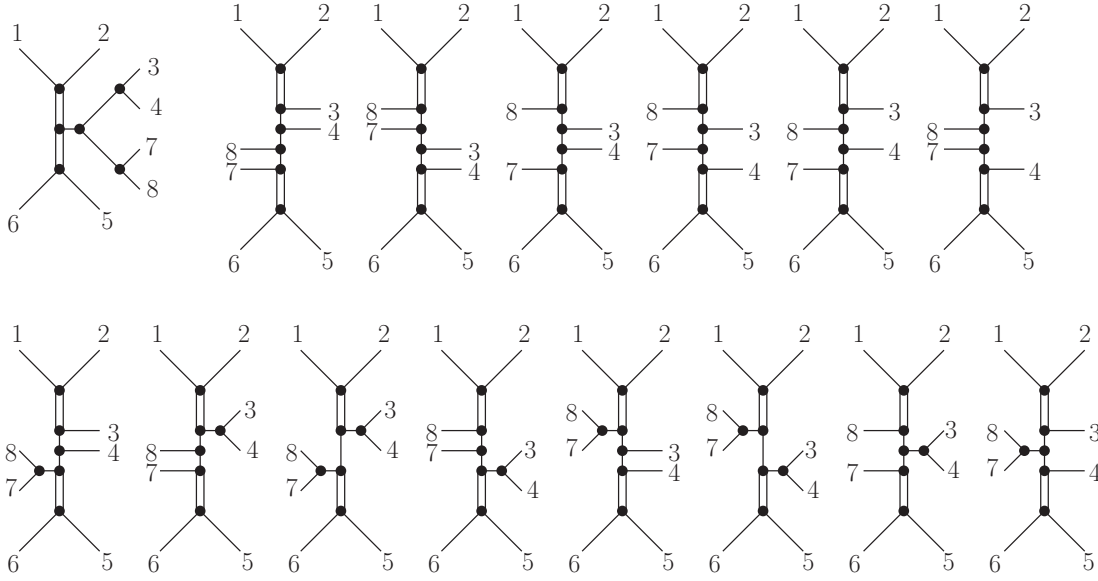


Figure 32. The 15 Feynman diagrams corresponding to the 4-regular graph figure 12.d.

Example figure 12.d: this is an eight-point example with CHY-integrand

$$\frac{1}{z_{12}^3 z_{56}^3 z_{34}^2 z_{78}^2 z_{23} z_{45} z_{67} z_{81} z_{47} z_{83}}. \quad (4.36)$$

The possible subsets for poles are $\{1, 2\}$, $\{5, 6\}$, $\{3, 4\}$, $\{7, 8\}$, $\{4, 5, 6\}$, $\{5, 6, 7\}$, $\{1, 2, 3\}$, $\{1, 2, 8\}$, $\{3, 4, 7, 8\}$, $\{1, 2, 3, 4\}$, $\{1, 2, 7, 8\}$, $\{4, 5, 6, 7\}$. From them we can construct 15 compatible combinations of subsets as

$$\begin{aligned} & \{\{1, 2\}, \{5, 6\}, \{3, 4, 7, 8\}, \{3, 4\}, \{7, 8\}\}, \\ & \{\{1, 2\}, \{5, 6\}, \{1, 2, 3, 4\}, [\{1, 2, 3\} \text{ or } \{3, 4\}], [\{7, 8\} \text{ or } \{5, 6, 7\}]\}, \\ & \{\{1, 2\}, \{5, 6\}, \{1, 2, 7, 8\}, [\{1, 2, 8\} \text{ or } \{7, 8\}], [\{3, 4\} \text{ or } \{4, 5, 6\}]\}, \\ & \{\{1, 2\}, \{5, 6\}, \{4, 5, 6, 7\}, [\{1, 2, 3\} \text{ or } \{1, 2, 8\}], [\{4, 5, 6\} \text{ or } \{5, 6, 7\}]\}, \\ & \{\{1, 2\}, \{5, 6\}, \{1, 2, 8\}, \{5, 6, 7\}, \{3, 4\}\}, \quad \{\{1, 2\}, \{5, 6\}, \{1, 2, 3\}, \{4, 5, 6\}, \{7, 8\}\}. \end{aligned}$$

The corresponding Feynman diagrams are presented in figure 32. So accordingly, the result can be written down by summing over all 15 Feynman diagrams, by applying Feynman rules $\mathcal{R}_{\text{ule}}^I$, $\mathcal{R}_{\text{ule}}^{III}$ to corresponding propagators, as

$$\begin{aligned} & \frac{1}{s_{34}s_{78}s_{3478}} \mathcal{R}_{\text{ule}}^{III}[\{1\}, \{2\}, \{3, 4, 7, 8\}, \{5\}, \{6\}] \\ & + \frac{1}{s_{123}s_{1234}s_{567}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3\}, \{4, 5, 6, 7, 8\}] \mathcal{R}_{\text{ule}}^I[\{7\}, \{8, 1, 2, 3, 4\}, \{5\}, \{6\}] \\ & + \frac{1}{s_{123}s_{1234}s_{78}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3\}, \{4, 5, 6, 7, 8\}] \mathcal{R}_{\text{ule}}^I[\{7, 8\}, \{1, 2, 3, 4\}, \{5\}, \{6\}] \\ & + \frac{1}{s_{34}s_{1234}s_{567}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3, 4\}, \{5, 6, 7, 8\}] \mathcal{R}_{\text{ule}}^I[\{7\}, \{8, 1, 2, 3, 4\}, \{5\}, \{6\}] \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{s_{1234}s_{34}s_{78}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3, 4\}, \{5, 6, 7, 8\}] \mathcal{R}_{\text{ule}}^I[\{7, 8\}, \{1, 2, 3, 4\}, \{5\}, \{6\}] \\
 & + \frac{1}{s_{1278}s_{128}s_{456}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3, 4, 5, 6, 7\}, \{8\}] \mathcal{R}_{\text{ule}}^I[\{7, 8, 1, 2, 3\}, \{4\}, \{5\}, \{6\}] \\
 & + \frac{1}{s_{1278}s_{34}s_{128}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3, 4, 5, 6, 7\}, \{8\}] \mathcal{R}_{\text{ule}}^I[\{7, 8, 1, 2\}, \{3, 4\}, \{5\}, \{6\}] \\
 & + \frac{1}{s_{1278}s_{78}s_{456}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3, 4, 5, 6\}, \{7, 8\}] \mathcal{R}_{\text{ule}}^I[\{7, 8, 1, 2, 3\}, \{4\}, \{5\}, \{6\}] \\
 & + \frac{1}{s_{1278}s_{78}s_{34}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3, 4, 5, 6\}, \{7, 8\}] \mathcal{R}_{\text{ule}}^I[\{7, 8, 1, 2\}, \{3, 4\}, \{5\}, \{6\}] \\
 & + \frac{1}{s_{4567}s_{128}s_{567}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3, 4, 5, 6, 7\}, \{8\}] \mathcal{R}_{\text{ule}}^I[\{7\}, \{8, 1, 2, 3, 4\}, \{5\}, \{6\}] \\
 & + \frac{1}{s_{4567}s_{128}s_{456}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3, 4, 5, 6, 7\}, \{8\}] \mathcal{R}_{\text{ule}}^I[\{7, 8, 1, 2, 3\}, \{4\}, \{5\}, \{6\}] \\
 & + \frac{1}{s_{4567}s_{123}s_{567}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3\}, \{4, 5, 6, 7, 8\}] \mathcal{R}_{\text{ule}}^I[\{7\}, \{8, 1, 2, 3, 4\}, \{5\}, \{6\}] \\
 & + \frac{1}{s_{4567}s_{123}s_{456}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3\}, \{4, 5, 6, 7, 8\}] \mathcal{R}_{\text{ule}}^I[\{7, 8, 1, 2, 3\}, \{4\}, \{5\}, \{6\}] \\
 & + \frac{1}{s_{128}s_{567}s_{34}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3, 4, 5, 6, 7\}, \{8\}] \mathcal{R}_{\text{ule}}^I[\{7\}, \{8, 1, 2, 3, 4\}, \{5\}, \{6\}] \\
 & + \frac{1}{s_{123}s_{456}s_{78}} \mathcal{R}_{\text{ule}}^I[\{1\}, \{2\}, \{3\}, \{4, 5, 6, 7, 8\}] \mathcal{R}_{\text{ule}}^I[\{7, 8, 1, 2, 3\}, \{4\}, \{5\}, \{6\}]. \quad (4.37)
 \end{aligned}$$

Example figure 12.e: this is a seven-point example with CHY-integrand

$$- \frac{1}{z_{71}^2 z_{12}^2 z_{34}^3 z_{56}^3 z_{72} z_{23} z_{45} z_{67}}. \quad (4.38)$$

The possible subsets for poles are $\{1, 2, 7\}$, $\{3, 4\}$, $\{5, 6\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$, $\{4, 5, 6\}$, $\{5, 6, 7\}$, $\{1, 7\}$, $\{1, 2\}$. From them we can construct nine compatible combinations of subsets as

$$\begin{aligned}
 & \{\{1, 2, 7\}, \{3, 4\}, \{5, 6\}, [\{1, 7\} \text{ or } \{1, 2\}]\}, \quad \{\{3, 4\}, \{5, 6\}, \{2, 3, 4\}, \{5, 6, 7\}\}, \\
 & \{\{3, 4\}, \{5, 6\}, \{2, 3, 4\}, \{1, 7\}\}, \quad \{\{3, 4\}, \{5, 6\}, \{5, 6, 7\}, \{1, 2\}\} \\
 & \{\{1, 2, 7\}, \{3, 4\}, \{3, 4, 5\}, [\{1, 7\} \text{ or } \{1, 2\}]\}, \quad \{\{1, 2, 7\}, \{5, 6\}, \{4, 5, 6\}, [\{1, 7\} \text{ or } \{1, 2\}]\}.
 \end{aligned}$$

The corresponding nine Feynman diagrams are shown in figure 33. Applying the rules, we simply get the result

$$\begin{aligned}
 & \frac{1}{s_{17}} \mathcal{R}_{\text{ule}}^{\text{IX}}[\{7, 1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}] + \frac{1}{s_{12}} \mathcal{R}_{\text{ule}}^{\text{IX}}[\{7\}, \{1, 2\}, \{3\}, \{4\}, \{5\}, \{6\}] \\
 & + \frac{1}{s_{17}s_{456}} \mathcal{R}_{\text{ule}}^I[\{7, 1\}, \{2\}, \{3\}, \{4, 5, 6\}] \mathcal{R}_{\text{ule}}^I[\{7, 1, 2, 3\}, \{4\}, \{5\}, \{6\}] \\
 & + \frac{1}{s_{17}s_{345}} \mathcal{R}_{\text{ule}}^I[\{3\}, \{4\}, \{5\}, \{5, 7, 1, 2\}] \mathcal{R}_{\text{ule}}^I[\{3, 4, 5\}, \{6\}, \{7, 1\}, \{2\}] \\
 & + \frac{1}{s_{17}s_{234}} \mathcal{R}_{\text{ule}}^I[\{5\}, \{6\}, \{7, 1\}, \{2, 3, 4\}] \mathcal{R}_{\text{ule}}^I[\{5, 6, 7, 1\}, \{2\}, \{3\}, \{4\}] \\
 & + \frac{1}{s_{12}s_{456}} \mathcal{R}_{\text{ule}}^I[\{7\}, \{1, 2\}, \{3\}, \{4, 5, 6\}] \mathcal{R}_{\text{ule}}^I[\{7, 1, 2, 3\}, \{4\}, \{5\}, \{6\}]
 \end{aligned}$$

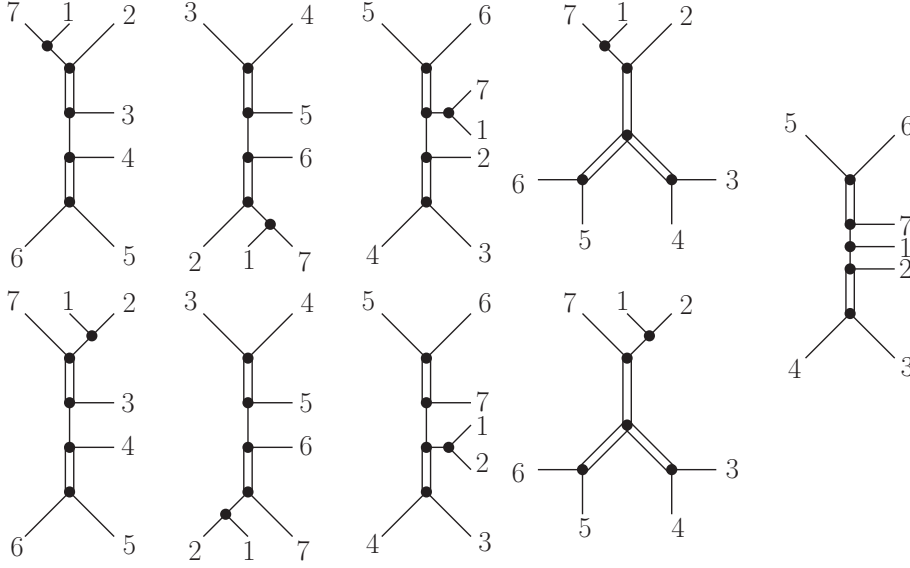


Figure 33. The nine Feynman diagrams corresponding to the 4-regular graph figure 12.e.

$$\begin{aligned}
 & + \frac{1}{s_{12}s_{345}} \mathcal{R}_{\text{ule}}^I[\{3\}, \{4\}, \{5\}, \{6, 7, 1, 2\}] \mathcal{R}_{\text{ule}}^I[\{3, 4, 5\}, \{6\}, \{7\}, \{1, 2\}] \\
 & + \frac{1}{s_{12}s_{567}} \mathcal{R}_{\text{ule}}^I[\{5\}, \{6\}, \{7\}, \{1, 2, 3, 4\}] \mathcal{R}_{\text{ule}}^I[\{5, 6, 7\}, \{1, 2\}, \{3\}, \{4\}] \\
 & + \frac{1}{s_{234}s_{567}} \mathcal{R}_{\text{ule}}^I[\{5\}, \{6\}, \{7\}, \{1, 2, 3, 4\}] \mathcal{R}_{\text{ule}}^I[\{5, 6, 7, 1\}, \{2\}, \{3\}, \{4\}]. \quad (4.39)
 \end{aligned}$$

Example figure 12.f: this is an eight-point example with CHY-integrand

$$- \frac{1}{z_{12}^2 z_{34}^2 z_{56}^3 z_{78}^3 z_{24} z_{45} z_{67} z_{81} z_{13} z_{32}}. \quad (4.40)$$

The possible subsets for poles are $\{1, 2, 3, 4\}$, $\{5, 6\}$, $\{7, 8\}$, $\{3, 4, 5, 6\}$, $\{4, 5, 6\}$, $\{5, 6, 7\}$, $\{6, 7, 8\}$, $\{7, 8, 1\}$, $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{1, 2\}$, $\{3, 4\}$. There are 15 compatible combinations of subsets from them, as

$$\begin{aligned}
 & \{\{1, 2, 3, 4\}, \{5, 6\}, \{7, 8\}, [\{1, 2, 3\}, \{1, 2\} \text{ or } \{2, 3, 4\}, \{3, 4\} \text{ or } \{1, 2\}, \{3, 4\}]\}, \\
 & \{\{1, 2, 3, 4\}, \{5, 6\}, \{5, 6, 7\}, [\{1, 2, 3\}, \{1, 2\} \text{ or } \{2, 3, 4\}, \{3, 4\} \text{ or } \{1, 2\}, \{3, 4\}]\}, \\
 & \{\{1, 2, 3, 4\}, \{7, 8\}, \{6, 7, 8\}, [\{1, 2, 3\}, \{1, 2\} \text{ or } \{2, 3, 4\}, \{3, 4\} \text{ or } \{1, 2\}, \{3, 4\}]\}, \\
 & \{\{5, 6\}, \{7, 8\}, \{3, 4, 5, 6\}, [\{3, 4\} \text{ or } \{4, 5, 6\}], [\{1, 2\} \text{ or } \{7, 8, 1\}]\}, \\
 & \{\{5, 6\}, \{7, 8\}, \{4, 5, 6\}, \{1, 2, 3\}, \{1, 2\}\}, \quad \{\{5, 6\}, \{7, 8\}, \{7, 8, 1\}, \{2, 3, 4\}, \{3, 4\}\}.
 \end{aligned}$$

The corresponding 15 Feynman diagrams are illustrated in figure 34. The result is trivially given by summing over all 15 Feynman diagrams with $\mathcal{R}_{\text{ule}}^I$, $\mathcal{R}_{\text{ule}}^{\text{IX}}$, and to save space we will not write it down explicitly here.

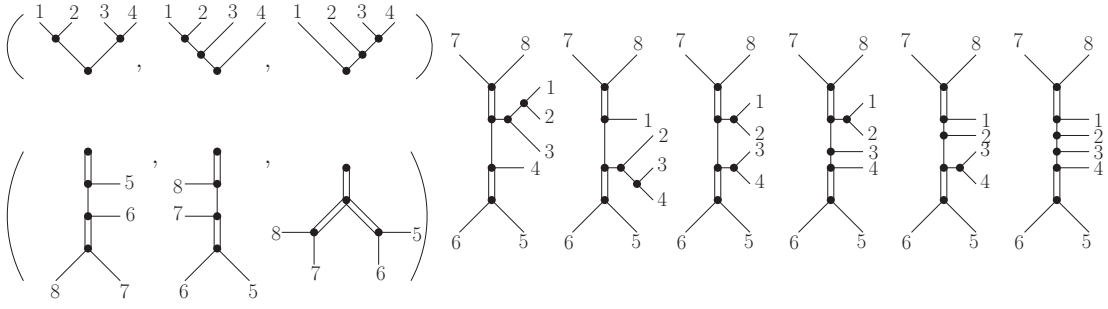


Figure 34. The 15 Feynman diagrams corresponding to the 4-regular graph figure 12.f.

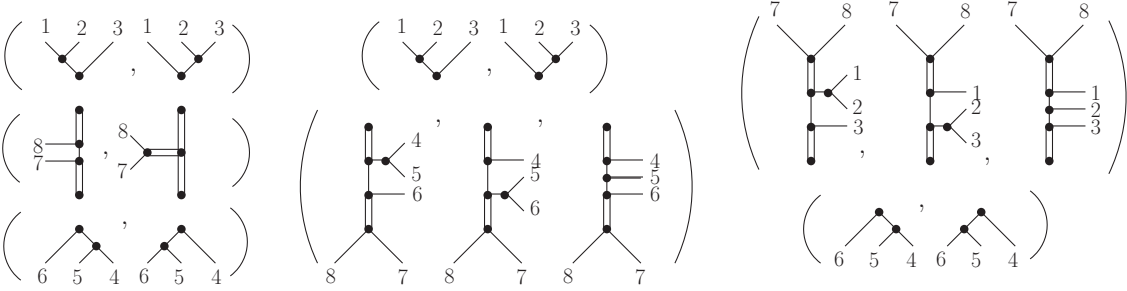


Figure 35. The 20 Feynman diagrams corresponding to the 4-regular graph figure 12.g.

Example figure 12.g: this is an eight-point example with CHY-integrand

$$\frac{1}{z_{12}^2 z_{23}^2 z_{45}^2 z_{56}^2 z_{78}^3 z_{13} z_{34} z_{46} z_{67} z_{81}}. \quad (4.41)$$

The possible subsets for poles are $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8\}$, $\{1, 2\}$, $\{2, 3\}$, $\{4, 5\}$, $\{5, 6\}$, $\{1, 7, 8\}$, $\{6, 7, 8\}$, $\{1, 2, 3, 4\}$, $\{1, 2, 3, 8\}$, $\{1, 2, 7, 8\}$. From them we can construct 20 compatible combinations of subsets as

$$\begin{aligned} & \{ \{1, 2, 3\}, \{4, 5, 6\}, \{7, 8\}, [\{1, 2\} \text{ or } \{2, 3\}], [\{4, 5\} \text{ or } \{5, 6\}] \}, \\ & \{ \{1, 2, 3\}, \{4, 5, 6\}, \{1, 2, 3, 8\}, [\{1, 2\} \text{ or } \{2, 3\}], [\{4, 5\} \text{ or } \{5, 6\}] \}, \\ & \{ \{1, 2, 3\}, \{7, 8\}, [\{1, 2\} \text{ or } \{2, 3\}], [\{4, 5\}, \{6, 7, 8\} \text{ or } \{5, 6\}, \{1, 2, 3, 4\} \text{ or } \{6, 7, 8\}, \{1, 2, 3, 4\}] \}, \\ & \{ \{4, 5, 6\}, \{7, 8\}, [\{4, 5\} \text{ or } \{5, 6\}], [\{1, 2\}, \{1, 2, 7, 8\} \text{ or } \{2, 3\}, \{1, 7, 8\} \text{ or } \{1, 7, 8\}, \{1, 2, 7, 8\}] \}. \end{aligned}$$

The corresponding 20 Feynman diagrams for this CHY-integrand is presented in figure 35. By applying Feynman rules on corresponding propagators, one immediately get the result, which we will not repeat here.

Example figure 12.h: this is a nine-point example with CHY-integrand

$$-\frac{1}{z_{12}^2 z_{23}^2 z_{45}^2 z_{56}^2 z_{78}^2 z_{89}^2 z_{13} z_{34} z_{46} z_{67} z_{79} z_{91}}. \quad (4.42)$$

The possible subsets for poles are $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9\}$, $\{1, 2\}$, $\{2, 3\}$, $\{4, 5\}$, $\{5, 6\}$, $\{7, 8\}$, $\{8, 9\}$, $\{1, 2, 3, 4\}$, $\{1, 2, 3, 9\}$, $\{1, 7, 8, 9\}$, $\{3, 4, 5, 6\}$, $\{4, 5, 6, 7\}$, $\{6, 7, 8, 9\}$. From

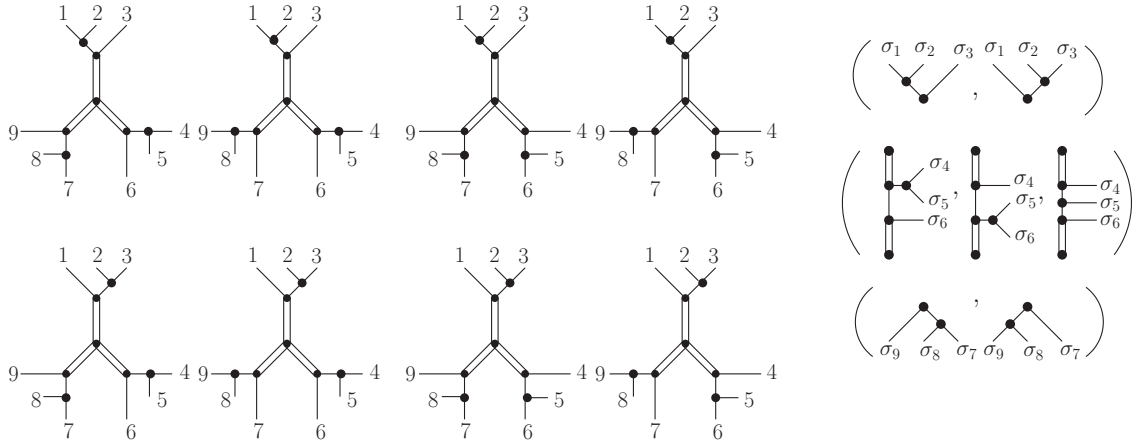


Figure 36. The 44 Feynman diagrams corresponding to the 4-regular graph figure 12.h. $\{\sigma_1, \dots, \sigma_9\}$ takes value of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $\{4, 5, 6, 7, 8, 9, 1, 2, 3\}$ and $\{7, 8, 9, 1, 2, 3, 4, 5, 6\}$.

them we can construct 44 compatible combinations of subsets as

$$\begin{aligned}
 & \{ \{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, [\{1, 2\} \text{ or } \{2, 3\}], [\{4, 5\} \text{ or } \{5, 6\}], [\{7, 8\} \text{ or } \{8, 9\}] \}, \\
 & \{ \{1, 2, 3\}, \{4, 5, 6\}, [\{1, 2\} \text{ or } \{2, 3\}], [\{4, 5\} \text{ or } \{5, 6\}], \\
 & \quad [\{7, 8\}, \{1, 2, 3, 9\} \text{ or } \{1, 2, 3, 9\}, \{4, 5, 6, 7\} \text{ or } \{8, 9\}, \{4, 5, 6, 7\}] \}, \\
 & \{ \{1, 2, 3\}, \{7, 8, 9\}, [\{1, 2\} \text{ or } \{2, 3\}], [\{7, 8\} \text{ or } \{8, 9\}], \\
 & \quad [\{4, 5\}, \{6, 7, 8, 9\} \text{ or } \{1, 2, 3, 4\}, \{6, 7, 8, 9\} \text{ or } \{5, 6\}, \{1, 2, 3, 4\}] \}, \\
 & \{ \{4, 5, 6\}, \{7, 8, 9\}, [\{4, 5\} \text{ or } \{5, 6\}], [\{7, 8\} \text{ or } \{8, 9\}], \\
 & \quad [\{1, 2\}, \{3, 4, 5, 6\} \text{ or } \{3, 4, 5, 6\}, \{1, 7, 8, 9\} \text{ or } \{2, 3\}, \{1, 7, 8, 9\}] \}.
 \end{aligned}$$

Then we can draw the corresponding 44 Feynman diagrams as in figure 36. When summing over all 44 Feynman diagrams with $\mathcal{R}_{\text{ule}}^{\text{I}}$, $\mathcal{R}_{\text{ule}}^{\text{IX}}$, it miraculously produces the correct result, which is confirmed numerically.

5 Discussion and conclusion

The *integration rule* presented in [18, 19] is quite efficient and elegant. As the word *rule* suggests, there is actually no practical computation but just pattern matching. But the *integration algorithm* is limited to CHY-integrands with simple poles therein. In order to compute CHY-integrands with higher-order poles, one need to use for example the Pfaffian identities. Although the Pfaffian identities can relate a CHY-integrand of higher-order poles with those of simple poles, the number of terms in the identities suffers from factorial increasing. Also generally one Pfaffian identity is not enough to completely decompose a CHY-integrand of higher-order poles into terms with only simple poles, which makes

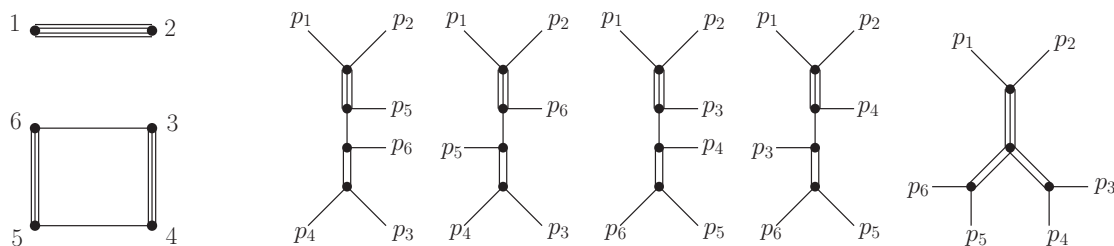


Figure 37. An example of CHY-integrand containing new pole structure with Feynman rule unknown.

computation involved. What is worse, some CHY-integrands, for example the one in figure 10.c, as already stated in [18], has no corresponding *3-regular* graph. So it can not be related to other CHY-integrand by Pfaffian identities at all.

In this paper, we sharpen the *integration algorithm* by providing Feynman rules for higher-order poles. With these rules, we can deal with CHY-integrands with corresponding higher-order poles exactly in the same spirit as in [18, 19]. For example again the CHY-integrand figure 10.c, previously not solved, can be instantly obtained by summing over nine Feynman diagrams of our Feynman rule $\mathcal{R}_{\text{ule}}^{\text{II}}$. Ample examples have been checked, not limited to those provided in section 4 of this paper, to support the validation of these rules. Again as the name *rule* suggests, if we forget the hard work spent in working out these rules, then the computation in fact involves no computation but just pattern matching, and all results come out instantly in Mathematica.

In a recent paper [20], Gomez proposed a so-called Λ -algorithm, to compute CHY-integrands with higher-order poles, in the framework of Λ scattering equations. The Λ -algorithm is quite elegant and general, but gauge-fixing is required during computation. One need to choose a proper gauge-fixing which involves no singular configurations, which is not always possible. As stated in [20], for example, the CHY-integrand figure 10.c, can only be computed with the help of general KLT construction. But with *integration algorithm* strengthened with Feynman rules of higher-order poles, all the examples therein, e.g., figures 3, 5, 10.b and even eight-point example 12.d, can be evaluated without effects. We remark that, in [20], $2P_i \cdot P_j$ is chosen to be the kinematic variables in Λ -algorithm but not s_{ij} , which is important for it to produce correct answer. In our Feynman rules of higher-order poles, we are forced to take the same choice. This would not be a coincidence, and we are eager to find out if there is any deep connection between these two approaches.

However, the *integration algorithm* also suffers from the problem that, although the higher-order poles have the standard structure of Feynman diagrams, they are *quasi-local*, and new pole structures would appear so that one need to create new rules for them. For example, when computing the CHY-integrand represented by *4-regular* graph as shown in figure 37, we end with five Feynman diagrams. The first four can be computed by using Feynman rules $\mathcal{R}_{\text{ule}}^{\text{I}}$, $\mathcal{R}_{\text{ule}}^{\text{II}}$ of double pole and triple pole, but for the last one, we need a new rule for the pole structure where a triple propagator and two double propagators are connected at one vertex. Since we have no generic method to *derive* the rule, it would be a problem for applying *integration algorithm* in most generic situations.

Based on current results of Feynman rules for higher-order poles, there are several directions worth to explore. The first and most important one is to derive these rules analytically. One could either achieve this in the CHY framework using the same method as in [18, 19], or use the newly proposed Λ -formalism in [20]. The latter one might be more convenient, since it naturally contains the pinching picture and iterative construction. The second considers the question that whether and how Feynman rules of different higher-order pole structures are related to each other. Although we have observed different Feynman rules for different higher-order poles, it would be very interesting to see how far would the *quasi-local* property preventing us from reducing all other Feynman rules of higher-order poles (e.g., $\mathcal{R}_{\text{ule}}^{\text{IX}}$), to those with only a single double or triple pole. Although as shown in [18], the Pfaffian identities⁸ can relate one higher-order pole to another, there might be other better approaches, and one need to figure out the details. Finally, with the Feynman rules of higher-order poles, one is ready to apply it in explicit theories and exploit details therein.

Acknowledgments

We would like to thank Song He, Gang Yang and Carlos Cardona for insightful discussions. This work is supported by Qiu-Shi Funding and the National Natural Science Foundation of China (NSFC) with Grant No.11135006, No.11125523 and No.11575156. RH would also like to acknowledge the supporting from Chinese Postdoctoral Administrative Committee.

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⁸There is a related question whether Pfaffian identities are complete or not, i.e., could they provide all independent relations among different pole structures so that all higher-order poles can be related to simple poles?

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